Large deviations and exit time asymptotics for diffusions and stochastic resonance

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Abstract

In this thesis, we study the asymptotic behavior of exit and transition times of certain weakly time inhomogeneous diffusion processes. Based on these asymptotics, a probabilistic notion of stochastic resonance is investigated. Large deviations techniques play the key role throughout this work.

In the first part we recall the large deviations theory for time homogeneous diffusions. Chapter 1 gives a brief overview of the classical results due to Freidlin und Wentzell. In Chapter 2 we present an extension of their theory to stochastic differential equations with locally Lipschitz coefficients that depend on the noise amplitude. Kramers' exit time law, which makes up the foundation stone for the results obtained in this thesis, is recalled in Chapter 3.

The second part deals with the phenomenon of stochastic resonance. That is, we study periodicity properties of diffusion processes. First, in Chapter 4 we explain the paradigm of stochastic resonance. Afterwards, physical notions of measuring periodicity properties of diffusion trajectories are discussed, and their drawbacks are pointed out. The latter suggest to follow an alternative probabilistic approach, which is introduced in Section 4.3 and discussed in subsequent chapters.
In Chapter 5 we derive a large deviations principle for diffusions subject to a weakly time dependent periodic drift term. Here the uniformity of the obtained large deviations bounds w.r.t. the system's parameters plays a key role for the treatment of transition time asymptotics in Chapter 6. These asymptotics represent the main result of the second part. They yield exact exponential transition rates, whose dependence on the time scale of the drift's period is given explicitly, thus allowing for a maximization of transition probabilities w.r.t. the time scale. This finally leads to the announced probabilistic notion of resonance, which is studied in Chapter 7.

In the third part we investigate the exit time asymptotics of a certain class of self-stabilizing diffusions. Diffusions of this type describe the limiting dynamics of interacting particle systems as the number of particles tends to infinity. In Chapter 8 we explain the connection between interacting particle systems and self-stabilizing diffusions. Moreover, we address the question of existence and uniqueness for self-stabilizing diffusions. The following Chapter 9 is devoted to the study of the large deviations behavior of these diffusions. In Chapter 10 Kramers' exit law is carried over to our class of self-stabilizing diffusions. Finally, the influence of self-stabilization is illustrated in Chapter 11.

Keywords:
large deviations, stochastic resonance, exit times, self-stabilizing diffusions

Zusammenfassung

Diese Arbeit behandelt die Asymptotik von Austritts- und Übergangszeiten für gewisse schwach zeitinhomogene Diffusionsprozesse. Darauf basierend wird ein probabilistischer Begriff der stochastischen Resonanz studiert. Hierbei spielen Techniken der großen Abweichungen eine zentrale Rolle.

Im ersten Teil werden Resultate aus der Theorie der großen Abweichungen für zeithomogene Diffusionen rekapituliert. Kapitel 1 enthält eine Zusammenfassung der klassischen Resultate von Freidlin und Wentzell. In Kapitel 2 werden Erweiterungen dieser Theorie auf stochastische Differentialgleichungen vorgestellt, deren Koeffizienten lokal Lipschitz sind und vom Rauschparameter abhängen. Im 3. Kapitel wird an das Kramers'sche Austrittszeitengesetz erinnert, das die wesentliche Grundlage für die in dieser Arbeit erzielten Ergebnisse bildet.

Der zweite Teil befasst sich mit dem Phänomen der stochastischen Resonanz, d.h. mit der Untersuchung von Periodizitätseigenschaften für Diffusionsprozesse. In Kapitel 4 wird zunächst das Paradigma der stochastischen Resonanz erklärt. Anschließend werden physikalische Gütemaße zur Messung der Periodizität von Diffusionstrajektorien diskutiert und deren Nachteile aufgezeigt. Letztere legen es nahe, einem alternativen, probabilistischen Ansatz zu folgen, der in Abschnitt 4.3 erläutert und in den anschließenden Kapiteln mathematisch behandelt wird.
Das 5. Kapitel dient der Herleitung eines Prinzips der großen Abweichungen für Diffusionen mit schwach zeitabhängigem, periodischem Drift. Hierbei spielt die Gleichmäßigkeit dieses Prinzips in allen Systemparametern eine wichtige Rolle für die Untersuchung der Asymptotik von Übergangszeiten in Kapitel 6. Diese Asymptotik bildet das zentrale Ergebnis des 2. Teils der Arbeit. Sie liefert exakte exponentielle Übergangsraten, deren Abhängigkeit von der Zeitskala der Periode des Drifts explizit gegeben ist. Hierdurch wird die Maximierung der Übergangswahrscheinlichkeiten bezüglich der Zeitskala ermöglicht, was schließlich zum in Kapitel 7 studierten probabilistischen Resonanzbegriff führt.

Teil drei der vorliegenden Arbeit setzt sich mit der Asymptotik der Austrittszeiten selbststabilisierender Diffusionen auseinander. Diffusionen dieses Typs beschreiben die Grenzdynamik interagierender Teilchensysteme im Limes unendlich vieler Teilchen. In Kapitel 8 wird zunächst der Zusammenhang zwischen interagierenden Teilchensystemen und selbststabilisierenden Diffusionen erläutert und die Existenz und Eindeutigkeit selbststabilisierender Diffusionen geklärt. Das folgende Kapitel 9 ist dem Studium der großen Abweichungen dieser Klasse von Diffusionen gewidmet. Im 10. Kapitel wird das Kramers'sche Austrittszeitengesetz auf selbststabilisierende Diffusionen übertragen. Schließlich wird der Einfluß der selbststabilisierenden Komponente auf das Austrittszeitengesetz in Kapitel 11 illustriert.

Schlagwörter:
grosse Abweichungen, stochastische Resonanz, Austrittszeiten, selbststabilisierende Diffusionen

Contents

Introduction

In this dissertation, we study exit time asymptotics for certain weakly time inhomogeneous diffusions. The natural mathematical tool for this purpose is large deviations theory for diffusion processes. In the first part we recall the classical Freidlin-Wentzell theory and extensions thereof as well as Kramers' exit time law for time homogeneous diffusion processes. These provide the basis for the subsequently treated large deviations principles and exit and transition times. The second part is devoted to the phenomenon of stochastic resonance. We employ a large deviations approach to analyze the asymptotic behavior of transition times between domains of attraction for weakly time inhomogeneous diffusions driven by a potential type drift term. In the third part we derive an analogue of Kramers' law for a class of self-stabilizing diffusions.

Part I – Large deviations for diffusions

The classical Freidlin-Wentzell theory ([48]) is concerned with the large deviations behavior of diffusions given by the autonomous SDE

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d, \tag{1}$$

where W is a Brownian motion. It describes the large deviations of the family $(X^\varepsilon)_{\varepsilon>0}$ from the solution of the deterministic system $\dot{\xi} = b(\xi),\ \xi_0 = x_0$, in the small noise limit $\varepsilon \to 0$. In its classical form, it is nowadays accessible in many textbooks, see e.g. [17], [19], and many others. We briefly review this theory in Chapter 1.

The typical assumption for Freidlin and Wentzell's classical theory consists of a global Lipschitz assumption on the coefficients of (1). Since the focus lies on 'potential type' SDEs in this thesis, i.e. the drift vector b resembles essentially the geometry of a potential gradient, such a global Lipschitz assumption is usually not fulfilled in our setting. For that reason we relax this assumption in Chapter 2 and at the same time allow for ε-dependent coefficients. Following Azencott's approach [2], we present an account of the Freidlin-Wentzell theory for the family of solutions of

$$dX_t^\varepsilon = b^\varepsilon(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\sigma^\varepsilon(X_t^\varepsilon)\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d\,. \tag{2}$$

Here the coefficients are assumed to be locally Lipschitz. It was shown in Azencott [2] and in two subsequent papers by Priouret [39] and Baldi et al. [3], who extended Azencott's approach to the full generality presented here, that the family of solutions of (2) satisfies a large deviations principle under rather broad conditions. If the vector fields b^ε and σ^ε converge locally uniformly to some locally Lipschitz limits b and σ, respectively, then the large deviations principle holds true. The approach provides a rather elegant proof, and – concerning regularity properties – requires the above mentioned local Lipschitz properties, and the existence of strong solutions to (2). To ensure the latter in our applications, we briefly recall some results about strong solvability of SDEs driven by a dissipative drift term. These are particularly well suited to potential type SDEs, and in the case of Brownian additive noise ($\sigma \equiv 1$) a strengthening of the dissipativity conditions guarantees that the diffusion is concentrated in compact sets with high probability.

The results obtained in this thesis are based on and motivated by Kramers' exit law for time homogeneous diffusions. It describes the large deviations asymptotics of the time a diffusive particle needs to leave a local attractor. For convenience, we recall this law in Chapter 3.

Part II – Transition times and stochastic resonance

The second part of this thesis deals with the problem of *stochastic resonance*, that is, we investigate periodicity properties of diffusion trajectories. Stochastic resonance is a natural phenomenon which transports the effect of *noise-induced amplification*. The key observation is that the addition of noise may expose or amplify periodicity properties of a (deterministic) system, and thus may result in an increase of signal quality. This somehow counterintuitive effect – the common sense of noise is that it destroys information – has been observed in many branches of natural sciences (see [21] for an overview). In Section 4.1 we explain the basic idea using a classical example from climatology due to Nicolis [36] and Benzi et al. [6].

Mathematically, the phenomenon of stochastic resonance was addressed only recently, and there are many open questions as of today. A prototypical example of a system that exhibits stochastic resonance is given by the randomly perturbed bistable system

$$dX_t^\varepsilon = -U'\Big(\frac{t}{T}, X_t^\varepsilon\Big)dt + \sqrt{\varepsilon}dW_t, \quad X_0^\varepsilon = x_0. \tag{3}$$

Here $U(t,\cdot)$ is a double-well potential that is periodic in time with period one, and W is a standard Brownian motion. To help intuition, one may think of the periodicity of U resulting from a periodic modulation of a symmetric double-well potential, for instance $U(t,x) = U_0(x) - Ax\sin\left(\frac{2\pi t}{T}\right)$ with $U_0(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$. In this setting, the diffusion X^ε turns out to be 'almost periodic' for certain parameter values of the period T and the noise amplitude ε. This effect is attributed to the noise, since the underlying deterministic system (i.e. (3) for $\varepsilon = 0$) is concentrated to the local attractors corresponding to U, hence the term stochastic resonance.

Freidlin [18] was the first to approach the question of periodicity mathematically. He showed that, in order to observe periodicity, T and ε must be linked via the relation $T \sim \exp\left\{\frac{\mu}{\varepsilon}\right\}$ as $\varepsilon \to 0$, where μ must exceed some treshold that originates in Kramers' law (see Section 4.2).

Towards a thorough description of stochastic resonance, one intends to describe the quality of periodicity. To optimize periodicity in ε and T, one needs to define *quality measures*, to quantify periodicity of diffusion trajectories, which goes beyond Freidlin's approach. Stochastic resonance is then understood in the sense of an *optimal tuning* w.r.t. the chosen quality measure, i.e. one has to determine the optimal relation between ε and T that maximizes periodicity w.r.t. this measure. Spectral quality measures such as the *spectral power amplification* and the *signal-to-noise ratio*, as well as information theoretic measures such as the entropy have a clear physical interpretation. But at least the spectral measures suffer from a serious defect. As proposed by McNamara and Wiesenfeld [33] in their theory of stochastic resonance, the resonance behavior of the diffusion should coincide with that of a simpler reduced process, namely a two-state Markov chain that mimics only the inter-well dynamics of the diffusion. However, as Pavlyukevich [38] pointed out (see Section 4.2), this reduction does not work for the spectral measures, i.e. they are not robust w.r.t. model reduction. For the entropy the question of robustness is unanswered as of today.

In Part II of this thesis, we propose a quality measure that was first suggested in the one-dimensional setting by Herrmann and Imkeller [23]. In its nature it is purely probabilistic, since it captures only the transisiton mechanism of the diffusion. Moreover, it overcomes the disadvantage of spectral measures, i.e. it is robust for the passage to the Markov chain. It does, however, not have such a clear physical interpretation, and might or might not be accepted by the physical community in the future.

In the general setting of finite dimensional diffusion processes, we provide a new approach to the probabilistic notion of optimal tuning introduced in [23] by systematically exploiting large deviations techniques. We consider the family of solutions to the SDE

$$dX_t^\varepsilon = b\Big(\frac{t}{T^\varepsilon}, X_t^\varepsilon\Big)\, dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0. \tag{4}$$

Here the time scale is chosen exponentially large, i.e. $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ for some $\mu > 0$, to allow for periodicity in accordance with Freidlin's result. The vector field b describes essentially the geometry induced by a double-well potential. More precisely, the deterministic system $\dot{\xi}_t = b(s, \xi_t)$ with *frozen* time parameter s is supposed to have two domains of attraction with equilibria x_- and x_+ that do not depend on $s \geq 0$. In the gradient case, where b is the spatial derivative of a time dependent potential, this corresponds to a two-well situation with time-invariant local minima $x_\pm$ and separatrix. The main object of study in Part II is the transition time

$$\tau_\varrho^\varepsilon = \inf\{t \geq 0 : \ X_t^\varepsilon \in B_\varrho(x_+)\}$$

at which the diffusion reaches a ϱ-neighborhood of x_+ when starting in the domain of attraction of x_-.

In order to obtain large deviations type asymptotics for this transition time, we carefully examine the large deviations behavior of X^ε in Chapter 5. Via comparison arguments, we prove a large deviations principle for (4) with exponential starting times. We show that the diffusion X^ε is exponentially equivalent to a time-homogeneous diffusion with a frozen drift $b(s, \cdot)$, where the time parameter s results from an appropriate scaling of the starting times. The crucial feature of the large deviations principle that makes up most of the technical complexity of the proof lies in the fact that the large deviations bounds are uniform w.r.t. all system parameters: the starting time, the initial condition and the scale parameter μ.

Based on the large deviations principle thus derived, we address the asymptotic behavior of the transition time τ_ϱ^ε in Chapter 6. Motivated by Kramers' law, we introduce the transition time

$$a_\mu = \inf\{s \geq 0 : \ e(s) \leq \mu\},$$

at which we expect the diffusion to undergo a transition in its natural time scale. Here $e(s)$ denotes the quasipotential of the time homogeneous diffusion associated to the drift $b(s, \cdot)$, i.e. $e(s)$ is the energy X^ε needs at time s to leave the domain of attraction of x_-. Using carefully chosen splittings of exponentially long time intervals allows us to employ large deviations techniques, to obtain asymptotic estimates for τ_ϱ^ε. Our main result (Theorem 6.3) states that for sufficiently small $\eta > 0$ and $h > 0$

$$\lim_{\varepsilon \to 0} \varepsilon \log \sup_{y \in B_\eta(x_-)} \mathbb{P}_y \left(\tau_\varrho \notin [(a_\mu - h)T^\varepsilon, (a_\mu + h)T^\varepsilon] \right) \ = \ \mu - e(a_\mu - h), \qquad (5)$$

i.e. the diffusion transits to the other domain of attraction within a small time window around $a_\mu T^\varepsilon$. This window is exponentially large, but arbitrarily small on the exponential scale.

In Chapter 7 we exploit these asymptotics to give the announced probabilistic notion of stochastic resonance. Due to obvious symmetry reasons, the asymptotic relation (5) holds similarly for transitions in the opposite direction. The explicit expression for the exponential rate of convergence in (5) allows us to perform a minmax optimization procedure w.r.t. μ, to maximize transitions between neighborhoods of meta-stable points in both directions, and thus to obtain an optimal time scale parameter. Moreover, we show that our optimal tuning coincides with the one of the corresponding Markov chain (see [24]), i.e. it allows for robust model reduction.

A completely different approach to SR was done by N. Berglund and B. Gentz in their papers [7, 10, 9, 8]. They use relaxation time estimates to obtain pathwise results on periodic behavior of randomly perturbed dynamical systems. In their context, periodicity comes essentially from the parametrization of the underlying deterministic system. The latter possesses bifurcation points through which the system is moved according to the chosen parametrization, either periodically or in hysteresis loops. A careful analysis of the random dynamical system shows that it follows the stable equilibrium of the deterministic system with a certain delay, which results in a 'pathwise' notion of stochastic resonance.

Part III – Large deviations and the exit problem for self-stabilizing diffusions

The third part of this thesis is devoted to an extension of the classical Kramers' law to a certain class of *self-stabilizing* diffusions. Such diffusions are obtained as meso-scopic limits of interacting particle systems, as the number of particles in an ensemble of identical ones tends to infinity. Motivated by empirical studies of stochastic resonance for interacting systems, we aim at proving a Kramers' type law for their low-dimensional approximation, which is the first step towards carrying over the probabilistic approach to stochastic resonance of Part II to these systems.

We study a class of diffusions governed by the d-dimensional SDE

$$dX_t^\varepsilon = V(X_t^\varepsilon)\,dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)\,du_t^\varepsilon(x)\,dt + \sqrt{\varepsilon}dW_t, \quad X_0^\varepsilon = x_0 \in \mathbb{R}^d. \qquad (6)$$

In this equation, V denotes a vector field on $\mathbb{R}^d$, which we think of as representing a potential gradient, and W is a Brownian motion. The second drift component involving the process' own law, $\mathcal{L}(X_t^\varepsilon) = u_t^\varepsilon$, introduces a feature that we call *self-stabilization*. The distance between the particle's instantaneous position X_t^ε and a fixed point x in state space is weighed by means of a so-called *interaction function* Φ and integrated in x against the law of X_t^ε itself. This effective additional drift can be seen as a measure for the average attractive force exerted on the particle by an independent copy of itself through the attractive potential Φ. It makes the diffusion inertial and stabilizes its motion in certain regions of the state space.

In Chapter 8 we explain the connection between interacting particle systems and self-stabilizing diffusions. Moreover, we clarify the question of existence and uniqueness for equation (6), and show that it admits unique strong and non-exploding solutions under reasonable geometric assumptions on V and Φ. Essentially, both the drift components involving V and Φ are dissipative, i.e. the particle's instantaneous drift decelerates its motion outside a large compact. Moreover $\Phi(x)$ is rotationally invariant and increasing in $\|x\|$ and $\Phi(0) = 0$, so that the second drift component indeed exerts an attractive force.

Having strong solutions at hand allows us to employ the usual large deviations tools. In Chapter 9 we derive a large deviations principle for the diffusion (6). It is seen, under our assumptions on the geometry of Φ, that the self-stabilizing diffusion is a small random perturbation of the deterministic system $\dot{\psi} = V(\psi)$ as in the 'classical' case without interaction, i.e. the self-stabilizing component changes the rate of convergence of the diffusion to its deterministic limit, but not the limit itself. Moreover, we prove an exponential approximation for X^ε in case the deterministic system ψ possesses a unique stable equilibrium point.

The exponential approximation is exploited in Chapter 10, to hook up to the theory of time homogeneous diffusions. Following the approach presented in Dembo and Zeitouni [17], we derive an analogue of Kramers' exit law for the diffusion X^ε under convexity assumptions on the geometry of V. The interaction function Φ is seen to act like an effective additional potential that may completely change the geometry described by the quasi-potential, which governs the exit law.

In order to illustrate the results of Chapter 10, we discuss the gradient case in Chapter 11, and provide some examples that illustrate the influence of self-stabilization.

For the sake of completeness, we summarize a few auxiliary results in two short appendices. Appendix A contains a version of Gronwall's lemma, which is used extensively in this work, as well as the exponential martingale inequality employed several times in Chapter 2. In Appendix B we provide some special large deviations results from the book by Dembo and Zeitouni [17], which are used in Chapter 10.

Declaration

Parts of the results obtained in this thesis have been published in The Annals of Applied Probability. The paper [27] mainly consists of the results of Chapters 5 to 7 and Section 2.1.2. The results of Part III are submitted for publication in the same journal, see [28].

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Berlin, May 13, 2007 Dierk Peithmann

Part I

Large deviations for diffusions

Chapter 1

Large deviations for time homogeneous diffusions: classical results

The large deviations behaviour of stochastic differential equations was first investigated by Freidlin and Wentzell [48] in their pioneering work of 1970. Since the beginning of the 1980s, various extensions and refinements have been established, and simplifications of their proofs have been pointed out.

In this chapter we shall recall the classical results due to Freidlin and Wentzell. We start with some generalities about large deviations.

1.1 The general setup

In this section we briefly recall some basic facts from large deviations theory. Let $(\mathcal{X}, \rho)$ be a Polish space, and let $\mathscr{B}$ be a σ-algebra over $\mathcal{X}$ that contains at least all Borel subsets (w.r.t. the topology generated by ρ). In our applications $\mathcal{X}$ will always be the space C_{0T} of continuous functions from $[0, T]$ to $\mathbb{R}^d$, endowed with the metric of uniform convergence.

1.1 Definition (Rate function)**.**

(i) A *rate function* is a lower semi continuous function $I : \mathcal{X} \to [0, \infty]$.
(ii) A *good rate function* is a function $I : \mathcal{X} \to [0, \infty]$ such that the level set $\{x \in \mathcal{X} : \ I(x) \leq \alpha\}$ is compact for each $\alpha \geq 0$.

Lower semicontinuity of I means that $\liminf_{n\to\infty} I(x_n) \geq I(x)$ whenever $x_n \to x$. Equivalently, the level set $\{I \leq \alpha\}$ is closed for each $\alpha > 0$.

1.2 Definition (Large deviations principle)**.** Let $I : \mathcal{X} \to [0, \infty]$ be a rate function on $\mathcal{X}$. A family $(\mu_\varepsilon)_{\varepsilon>0}$ of probability measures on $(\mathcal{X}, \mathscr{B})$ satisfies a large deviations principle (LDP) with rate function I if

$$-\inf_{x\in B^o} I(x) \;\le\; \liminf_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(B) \;\le\; \limsup_{\varepsilon\to 0} \varepsilon \log \mu_\varepsilon(B) \;\le\; -\inf_{x\in \bar{B}} I(x) \tag{1.1}$$

holds true for all $B \in \mathscr{B}$. Here B^o and $\bar{B}$ denote the interior and the closure of B, respectively.

Since the inner inequality in (1.1) holds trivially, the LDP is equivalent to demanding that the left inequality holds for all open sets and the right one for all closed sets. (This is only true as long as $\mathscr{B}$ contains all Borel sets.) It is easily seen that the LDP for a family of measures determines the associated rate function uniquely (see [17], Lemma 4.1.4).

In our applications we are interested in the large deviations of continuous stochastic processes, i.e. of random variables taking values in the space of continuous paths. Let us reformulate the definition in terms of random variables. Fix a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, and for each $\varepsilon > 0$ let X^ε be a $(\mathcal{X}, \mathscr{B})$-valued random variable. We say that the family of random variables $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle with rate function I if the measures $\mu_\varepsilon := \mathbb{P} \circ (X^\varepsilon)^{-1}$, $\varepsilon > 0$, satisfy the LDP.

To summarize, the LDP for the family $(X^\varepsilon)_{\varepsilon>0}$ means that for each closed set $F \subset \mathcal{X}$ we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \;\le\; -\inf_{x\in F} I(x), \tag{1.2}$$

and for each open set $G \subset \mathcal{X}$

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \;\ge\; -\inf_{x\in G} I(x). \tag{1.3}$$

We give a useful characterization of these two conditions (see [19], Theorem 3.3).

1.3 Proposition. *Let $I : \mathcal{X} \to [0, \infty]$ be a rate function, and let $(X^\varepsilon)_{\varepsilon>0}$ be a family of random variables taking values in $(\mathcal{X}, \mathscr{B})$.*

a) Condition (1.2) *implies the following condition:*
For each $\delta > 0$, $\gamma > 0$ and $\alpha > 0$ there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}\Big[\rho(X, \{I \le \alpha\}) \ge \delta\Big] \;\le\; \exp\Big\{-\tfrac{1}{\varepsilon}(\alpha - \gamma)\Big\} \quad \forall \varepsilon \le \varepsilon_0. \tag{1.4}$$

If I is a good rate function the converse also holds true, i.e. (1.4) *implies* (1.2)*.*

b) Condition (1.3) *is equivalent to:*
For each $\delta > 0$, $\gamma > 0$ and $x \in \mathcal{X}$ there exists $\varepsilon_0 > 0$ such that

$$\mathbb{P}\Big[\rho(X^\varepsilon, x) < \delta\Big] \;\ge\; \exp\Big\{-\tfrac{1}{\varepsilon}(I(x) + \gamma)\Big\} \quad \forall \varepsilon \le \varepsilon_0. \tag{1.5}$$

Proof.

a) Fix $\delta > 0$ and consider the set $F := \{x \in \mathcal{X} : \ \rho(x, \{I \leq \alpha\}) \geq \delta\}$. Since $\{I \leq \alpha\}$ is closed, the mapping $x \mapsto \rho(x, \{I \leq \alpha\})$ is continuous, so F is a closed set, and we have $I(x) > \alpha$ for $x \in F$ by its definition. Hence by (1.2)

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(\rho(X^\varepsilon, \{I \leq \alpha\}) \geq \delta\right) &= \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in F) \\ &\leq -\inf_{x \in F} I(x) \ \leq \ -\alpha, \end{aligned}$$

which yields (1.4).
For the converse, assume that I is a good rate function, and let $F \subset \mathcal{X}$ be closed. Fix $\gamma > 0$ such that $\alpha := \inf_{x \in F} I(x) - \gamma > 0$. Then $\{I \leq \alpha\} \cap F = \emptyset$, which implies that $\delta := \rho(\{I \leq \alpha\}, F)$ is strictly positive, since the continuous function $x \mapsto \rho(x, F)$ attains its minimum on the compact $\{I \leq \alpha\}$. Consequently, $F \subset \{x \in \mathcal{X} : \ \rho(x, \{I \leq \alpha\}) \geq \delta\}$, and (1.4) yields

$$\mathbb{P}\left[X^\varepsilon \in F\right] \ \leq \ \mathbb{P}\left[\rho(X^\varepsilon, \{I \leq \alpha\}) \geq \delta\right] \ \leq \ \exp\left\{ -\tfrac{1}{\varepsilon}(\alpha - \gamma)\right\}$$

for $\varepsilon \leq \varepsilon_0$, which by choice of α implies (1.2).

b) Condition (1.5) follows easily from (1.3) by letting $G = B_\delta(x) = \{y \in \mathcal{X} : \rho(y, x) < \delta\}$.
On the other hand, if (1.5) is satisfied, let $G \subset \mathcal{X}$ be an open set, and let $x \in G$ and $\delta > 0$ such that $B_\delta(x) \subset G$. Then by (1.5)

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in G) \ \geq \ \liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(X^\varepsilon \in B_\delta(x)) \ \geq \ -I(x),$$

and taking the supremum over $x \in G$ establishes (1.3). □

1.2 The Freidlin-Wentzell theory

In this chapter we recall the classical results of Freidlin and Wentzell. For a more detailed account of the following well known theory see [17] or [19].
We consider the family of $\mathbb{R}^d$-valued processes X^ε, $\varepsilon > 0$, defined by

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\sigma(X_t^\varepsilon)\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d, \tag{1.6}$$

on a fixed time interval $[0, T]$, where b and σ are Lipschitz continuous, and W is a d-dimensional Brownian motion. The large deviations principle for X^ε describes the exponential rate of convergence of rare events associated with X^ε in the small noise limit, i.e. as $\varepsilon \to 0$. The diffusion X^ε is a small random perturbation of the deterministic system

$$\dot{\xi} = b(\xi), \quad \xi_0 = x_0, \tag{1.7}$$

and a *large* deviation deals with the probability $\mathbb{P}(X^\varepsilon \in A)$ of an event A that satisfies $\sup_{t\in[0,T]} |\varphi_t - \xi_t| > \delta$ for some strictly positive δ and all $\varphi \in A$, i.e. A does not intersect the δ-tube around ξ.

The solution X^ε on the time interval $[0,T]$ is a $(\mathcal{X}, \mathscr{B})$-valued random variable, where $\mathcal{X} = C_{0T} := C([0,T], \mathbb{R}^d)$, and $\mathscr{B}$ denotes the Borel σ-field generated by the uniform metric

$$\rho_{0T}(\varphi, \psi) := \sup_{0\leq t\leq T} \|\varphi_t - \psi_t\|\,, \quad \varphi, \psi \in C_{0T}.$$

The intuitive idea to understand the large deviations of X^ε consists in regarding X^ε as a functional S of the scaled Brownian motion $\sqrt{\varepsilon}\,W$ and to use the contraction principle. This argument is rigorously correct only if the functional S is continuous, for instance if $\sigma \equiv 1$.
In general S will not be continuous, and the contraction principle is not directly applicable. Nevertheless, the formal result remains true also in this case, and one may argue as if S were continuous in order to derive the rate function.

Let us make these ideas precise. In a first step we have to describe the large deviations of the family of scaled Brownian motions. We denote by $H_{0T}^{x_0}$ the Cameron-Martin space of absolutely continuous functions with square integrable derivatives starting at x_0, i.e.

$$H_{0T}^{x_0} := \left\{ f : [0,T] \to \mathbb{R}^d \,\middle|\, f(t) = x_0 + \int_0^t g(s)\,ds \ \text{ for some } \ g \in L^2([0,T]) \right\}.$$

1.4 Theorem (Schilder)**.** *The family of scaled Brownian motions $(\sqrt{\varepsilon}W)_{\varepsilon>0}$ satisfies a large deviations principle in C_{0T} with good rate function*

$$J_{0T}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t\|^2\,dt, & \text{if } \varphi \in H_{0T}^0, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.8}$$

For a proof see [17], section 5.2. Let us next define the functional that maps paths of $\sqrt{\varepsilon}W$ to paths of the solution of (1.6). For $f \in H_{0T}^0$ denote by $g = S^{x_0}(f)$ the solution of the ODE

$$\dot{g}_t = b(g_t) + \sigma(g_t)\dot{f}_t, \quad g_0 = x_0, \tag{1.9}$$

which exists on $[0,T]$ due to the Lipschitz assumptions on b and σ. If $\sigma \equiv 1$, we may integrate this equation and extend the domain of S^{x_0} to C_{0T}. In that case we have $X^\varepsilon = S^{x_0}(\sqrt{\varepsilon}W)$, the mapping S^{x_0} is continuous, and we may apply the contraction principle in order to derive an LDP for X^ε from Schilder's theorem.

The following theorem shows that the application of the contraction principle is formally correct also in the general case.

1.5 Theorem (Freidlin and Wentzell). *The family of solutions $(X^\varepsilon)_{\varepsilon>0}$ satisfies a large deviations principle in C_{0T} with good rate function*

$$I_{0T}^{x_0}(\varphi) = \inf\Big\{ J_{0T}(f) :\ f \in H_{0T}^0, S^{x_0}(f) = \varphi \Big\}. \tag{1.10}$$

A proof of this result may be found in section 5.6 of [17] and in the next chapter under more general assumptions. In (1.10), the rate function of the Brownian motion is evaluated over all pre-images w.r.t. the mapping S^{x_0}. Under nondegeneracy assumptions there is only one pre-image, and the infimum may be evaluated further. If σ is invertible and $a := \sigma\sigma^*$ is uniformly positive definite, then $S^{x_0}(f) = \varphi$ implies that

$$\|\dot{f}_t\|^2 = \|\sigma^{-1}(\varphi_t)[\dot{\varphi}_t - b(\varphi_t)]\|^2 = [\dot{\varphi}_t - b(\varphi_t)]^* a^{-1}(\varphi_t)[\dot{\varphi}_t - b(\varphi_t)],$$

and therefore

$$I_{0T}^{x_0}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T [\dot{\varphi}_t - b(\varphi_t)]^* a^{-1}(\varphi_t)[\dot{\varphi}_t - b(\varphi_t)]\, dt, & \text{if } \varphi \in H_{0T}^{x_0}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.11}$$

For $\sigma \equiv 1$ this simplifies further, and we obtain

$$I_{0T}^{x_0}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t - b(\varphi_t)\|^2\, dt, & \text{if } \varphi \in H_{0T}^{x_0}, \\ +\infty, & \text{otherwise.} \end{cases} \tag{1.12}$$

Observe that $I_{0T}^{x_0}(\varphi) = 0$ means that φ (up to time T) coincides with the solution ξ of the deterministic equation (1.7), so $I_{0T}^{x_0}(\varphi)$ of (1.12) is essentially the L^2-deviation of φ from ξ.

Chapter 2

Large deviations for time homogeneous diffusions: extensions and refinements

The classical LDP due to Freidlin and Wentzell requires global Lipschitz conditions which are typically imposed in standard existence and uniqueness theorems for stochastic differential equations. In the setting of diffusions with a potential gradient type drift term the coefficients will not be globally Lipschitz. We therefore present a detailed account of an extension of the Freidlin-Wentzell theory which relaxes these assumptions, and additionally allows for ε-dependent coefficients. It is based on the papers by Priouret [39] and Baldi et al. [3], which in turn were motivated by Azencott's lecture notes [2].
The approach requires the existence of strong solutions. To ensure strong solvability in subsequent chapters, we start with some preliminaries on SDEs with locally Lipschitz coefficients in the next section. Section 2.2 is devoted to the large deviations principle.

2.1 SDEs with dissipative drift

This section provides some results on SDEs with locally Lipschitz coefficients and dissipative drift terms. Roughly speaking, dissipativity means in this context that the drift decelerates the diffusion outside a large compact, and thus ensures non-explosion. In the case of additive Brownian noise, we show that a superlinear dissipative drift guarantees the diffusion to be concentrated on compact sets with high probability.

2.1.1 Existence of strong solutions

In this subsection we recall some classical results concerning the existence of a strong solution for the non-autonomous d-dimensional SDE

$$dX_t = b(t, X_t)dt + \sigma(t, X_t)dW_t \tag{2.1}$$

driven by a d-dimensional Brownian motion W. We will stick to the standing assumption that the coefficients are locally Lipschitz, so pathwise uniqueness is guaranteed, and the question of existence of strong solutions amounts to the question of explosion.

The next proposition is a special case of [44], Theorem 10.2.2.

2.1 Proposition. *Let b and σ be locally Lipschitz. Assume that for each $T > 0$ there exists some $C_T > 0$ such that*

$$\|\sigma(t,x)\sigma^*(t,x)\| \le C_T(1 + \|x\|^2) \qquad \textit{and} \qquad \langle x, b(t,x)\rangle \le C_T(1 + \|x\|^2)$$

for $0 \le t \le T$, $x \in \mathbb{R}^d$.
Then (2.1) has a strong solution on $[0, \infty)$.

The following corollary will be particularly useful later on.

2.2 Corollary. *Let $\sigma \in \mathbb{R}^{d\times d}$ be a constant matrix, and let $b : \mathbb{R}_+ \times \mathbb{R}^d \to \mathbb{R}^d$, $(t,x) \mapsto b(t,x)$, be locally Lipschitz, uniformly w.r.t. $t \in [0,T]$ for each $T > 0$, and assume that*

$$\sup_{0\le t\le T} \|b(t,0)\| < \infty$$

for all $T > 0$. Moreover, suppose that there exists $r_0 > 0$ such that

$$\langle x, b(t,x)\rangle \le 0 \qquad \textit{for } \|x\| \ge r_0. \tag{2.2}$$

Then the SDE

$$dX_t = b(t, X_t)\,dt + \sigma dW_t$$

admits a unique strong solution for any random initial condition X_0.

2.1.2 Superlinear growth

In this subsection we shall exploit a strengthening of the dissipativity condition on the drift in Corollary 2.2, to show that the corresponding diffusions driven by Brownian noise of small amplitude stay in compact sets with high probability.

We consider the family $(X^\varepsilon)_{\varepsilon>0}$ defined by the SDE

$$dX_t^\varepsilon = b(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}dW_t, \quad X_0^\varepsilon = x_0 \in \mathbb{R}^d,$$

and suppose that b satisfies the assumptions of Corollary 2.2 with condition (2.2) replaced by the following stronger condition:

There are constants η, $R_0 > 0$ such that

$$\langle x, b(t,x)\rangle < -\eta \,\|x\| \qquad \text{for } t \geq 0 \text{ and } \|x\| \geq R_0. \tag{2.3}$$

We are interested in the small noise behavior of the exit time

$$\tau_R^\varepsilon := \inf\Big\{t \geq 0 : \ \|X_t^\varepsilon\| \geq R\Big\}, \quad R > 0.$$

The following theorem provides an asymptotic bound for τ_R^ε, which essentially states that X^ε stays inside $B_R(0)$ with very high probability, i.e. the probability of leaving $B_R(0)$ is exponentially small. The arguments of its proof are borrowed from the framework of self-attracting diffusions, see [40] and [25].

2.3 Theorem. *Let $\delta > 0$, and let $r : (0,\delta) \to (0,\infty)$ be a function satisfying*

$$\lim_{\varepsilon\to 0} \frac{\varepsilon}{r(\varepsilon)} = 0. \tag{2.4}$$

There exist universal constants $R_1, \varepsilon_1 > 0$ and $C > 0$ such that for $R \geq R_1$, $\varepsilon \leq \varepsilon_1$

$$\mathbb{P}_{x_0}\Big[\tau_R^\varepsilon \leq r(\varepsilon)\Big] \ \leq \ C\eta^2\, \frac{r(\varepsilon)}{\varepsilon}\, e^{-\frac{\eta R}{\varepsilon}} \qquad \textit{for} \quad \|x_0\| \leq \frac{R}{2}. \tag{2.5}$$

Here $\mathbb{P}_{x_0}$ indicates the law of X^ε starting at x_0.

2.4 Remark. The constants $R_1, \varepsilon_1 > 0$ and $C > 0$ are universal in the sense that they do not depend on the particular choice of the drift b, but only on the parameters η and R_0 introduced in the growth condition (2.3), and of course on the function $r(\varepsilon)$. Hence the bound (2.5) is uniform in the class of all diffusions that satisfy (2.3).

Proof of Theorem 2.3. For notational convenience, we suppress the superscript ε in X^ε, τ_R^ε etc. Choose a C^2-function $h : \mathbb{R}^d \to \mathbb{R}$ such that

$$\begin{cases} h(x) = \|x\| & \text{for } \|x\| \geq R_0, \\ h(x) \leq R_0 & \text{for } \|x\| \leq R_0, \end{cases}$$

where R_0 is the constant given by (2.3). By Itô's formula we have

$$h(X_t) = h(x_0) + \sqrt{\varepsilon}\int_0^t \Big\langle \nabla h(X_s),\, dW_s\Big\rangle + \int_0^t \Big\langle \nabla h, b\big(s,\cdot\big)\Big\rangle(X_s)\, ds + \frac{\varepsilon}{2}\int_0^t \triangle h(X_s)\, ds.$$

Let $\xi_t := \int_0^t \|\nabla h(X_s)\|^2\, ds$, i.e. ξ_t is the quadratic variation of the continuous local martingale $M_t := \int_0^t \nabla h(X_s)\, dW_s$, $t \geq 0$. Since $\nabla h(x) = \frac{x}{\|x\|}$ for $\|x\| \geq R_0$, we have

$d\xi_t = dt$ on $\{\|X_t\| \geq R_0\}$. Now we introduce an auxiliary process Z which shall serve to control $\|X\|$.

Let $0 < \tilde{\eta} < \eta$. According to Skorokhod's lemma (see Revuz, Yor [41]) there exists a unique pair of continuous adapted processes (Z, L) such that $Z \geq R_0$ and

$$Z = R_0 \vee \|x_0\| + \sqrt{\varepsilon} M - \tilde{\eta}\xi + L,$$

where L is an increasing process that increases only at times t for which $Z_t = R_0$. We will prove that

$$\|X_t\| \leq Z_t \quad \text{a.s. for all} \quad t \geq 0. \tag{2.6}$$

For that purpose, choose $f \in C^2(\mathbb{R})$ such that

$$\begin{cases} f(x) > 0 \text{ and } f'(x) > 0 \quad \text{for all} \quad x > 0, \\ f(x) = 0 \quad \text{for all} \quad x \leq 0. \end{cases}$$

According to Itô's formula, for $t \geq 0$

$$\begin{aligned} f(h(X_t) - Z_t) &= f(h(x_0) - \|x_0\| \vee R_0) + \int_0^t f'(h(X_s) - Z_s)\, d(h(X) - Z))_s \\ &\quad + \frac{1}{2}\int_0^t f''(h(X_s) - Z_s)\, d\langle h(X) - Z\rangle_s. \end{aligned}$$

By definition of h and Z we have $h(X_t) \leq Z_t$ on $\{\|X_t\| \leq R_0\}$, so $\{h(X_t) > Z_t\} = \{\|X_t\| > Z_t\}$. Moreover, by definition of Z, $h(X) - Z$ is a finite variation process. Hence the expression

$$\int_0^t f'\Big(\|X_s\| - Z_s\Big) \left\{ \frac{1}{\|X_s\|}\Big\langle X_s, b\Big(s, X_s\Big)\Big\rangle + \frac{\varepsilon}{2}\triangle h(X_s) + \tilde{\eta} \right\} ds - \int_0^t f'(\|X_s\| - Z_s)\, dL_s$$

is an upper bound of $f(h(X_t) - Z_t)$. Furthermore, $\triangle h(x) = \frac{d-1}{\|x\|}$ for $\|x\| \geq R_0$, which by (2.3) implies

$$\frac{1}{\|X_s\|}\Big\langle X_s, b\Big(s, X_s\Big)\Big\rangle + \frac{\varepsilon}{2}\triangle h(X_s) + \tilde{\eta} \;<\; \frac{\varepsilon(d-1)}{2\,\|X_s\|} + \tilde{\eta} - \eta \quad \text{on } \{\|X_s\| > Z_s\}.$$

The latter expression is negative if ε is small enough, so we may find $\varepsilon_0 > 0$ such that $f(\|X_t\| - Z_t) \leq 0$ for $\varepsilon < \varepsilon_0$. This implies $\|X_t\| \leq Z_t$ a.s. by definition of f, and (2.6) is established.

The inequality (2.6) allows us to bound the exit probability of X by that of Z. If Q denotes the law of the process Z, we see that for any $\alpha > 0$

$$\mathbb{P}_{x_0}\Big[\tau_R \leq r(\varepsilon)\Big] \;\leq\; Q\Big[\tau_R \leq r(\varepsilon)\Big] \;\leq\; e^{\alpha r(\varepsilon)}\, \mathbb{E}_Q\Big[e^{-\alpha\tau_R}\Big]. \tag{2.7}$$

In order to find a bound for the right hand side of (2.7), let $K := \sup_{\|x\| \le R_0} \|\nabla h(x)\|^2$. Then we have $\xi_t \le Kt$ for all $t \ge 0$. Note that w.l.o.g. h can be chosen such that $K \le 2R_0$. Now observe that, by Itô's formula, for any $\phi \in C^2(\mathbb{R})$

$$\begin{aligned} d\Big(\phi(Z_t)\, e^{-\frac{\alpha}{K}\xi_t}\Big) &= \sqrt{\varepsilon}\, \phi'(Z_t) e^{-\frac{\alpha}{K}\xi_t}\, dM_t + \phi'(Z_t)\, e^{-\frac{\alpha}{K}\xi_t}\, dL_t \\ &\quad + e^{-\frac{\alpha}{K}\xi_t} \Big\{ \frac{\varepsilon}{2}\phi''(Z_t) - \tilde\eta \phi'(Z_t) - \frac{\alpha}{K}\phi(Z_t) \Big\}\, d\xi_t. \end{aligned}$$

Let $R \ge R_0$. If we choose ϕ such that

$$\begin{cases} \frac{\varepsilon}{2}\phi''(y) - \tilde\eta\phi'(y) - \frac{\alpha}{K}\phi(y) = 0 & \text{for } y \in [R_0, R], \\ \phi'(R_0) = 0, \quad \phi(R) = 1, \end{cases}$$

then $\phi(Z_t)e^{-\frac{\alpha}{K}\xi_t}$ is a local martingale which is bounded up to time τ_R. By the stopping theorem we obtain

$$\phi(R_0 \vee \|x_0\|) = \mathbb{E}_Q\Big[\phi(Z_{\tau_R}) e^{-\frac{\alpha}{K}\xi_{\tau_R}}\Big] = \mathbb{E}_Q\Big[e^{-\frac{\alpha}{K}\xi_{\tau_R}}\Big]. \tag{2.8}$$

But $\xi_{\tau_R} \le K\tau_R$, which implies $\mathbb{E}_Q\Big[e^{-\frac{\alpha}{K}\xi_{\tau_R}}\Big] \ge \mathbb{E}_Q\Big[e^{-\alpha\tau_R}\Big]$, and we deduce from (2.7) that

$$\mathbb{P}_{x_0}\Big[\tau_R \le r(\varepsilon)\Big] \;\le\; e^{\alpha r(\varepsilon)}\, \mathbb{E}_Q\Big[e^{-\frac{\alpha}{K}\xi_{\tau_R}}\Big] \;=\; e^{\alpha r(\varepsilon)} \phi(R_0 \vee \|x\|). \tag{2.9}$$

Solving the differential equation for ϕ yields

$$\phi(x) = \frac{-\lambda^- e^{\lambda^+(x-R_0)} + \lambda^+ e^{\lambda^-(x-R_0)}}{-\lambda^- e^{\lambda^+(R-R_0)} + \lambda^+ e^{\lambda^-(R-R_0)}}, \quad \text{where } \lambda^\pm = \frac{1}{\varepsilon}\left(\tilde\eta \pm \sqrt{\tilde\eta^2 + 2\frac{\alpha}{K}\varepsilon}\right).$$

Thus

$$\phi(x) \le \frac{(\lambda^+ - \lambda^-)\, e^{\lambda^+(x-R_0)}}{(-\lambda^-)\, e^{\lambda^+(R-R_0)}},$$

and taking $\alpha = r(\varepsilon)^{-1}$ in (2.9) yields

$$\mathbb{P}_{x_0}\Big[\tau_R \le r(\varepsilon)\Big] \;\le\; \exp(1)\phi(R_0 \vee \|x_0\|) \;\le\; \frac{\lambda^+ - \lambda^-}{-\lambda^-} \exp\Big\{1 + \lambda^+(R_0 \vee \|x_0\| - R)\Big\}.$$

It is obvious that $\exp\Big\{\lambda^+(R_0 \vee \|x_0\| - R)\Big\} \le \exp\Big\{-\frac{\tilde\eta R}{\varepsilon}\Big\}$ for $R \ge 2(\|x_0\| \vee R_0)$, so it remains to comment on the prefactor. We have

$$\frac{\lambda^+ - \lambda^-}{-\lambda^-} = \frac{2\sqrt{\tilde\eta^2 + 2\frac{\alpha}{K}\varepsilon}}{\sqrt{\tilde\eta^2 + 2\frac{\alpha}{K}\varepsilon} - \tilde\eta} \le \frac{4\Big(\tilde\eta^2 + \frac{2\varepsilon}{Kr(\varepsilon)}\Big)}{\frac{2\varepsilon}{Kr(\varepsilon)}}.$$

Since $\lim_{\varepsilon\to 0} \frac{\varepsilon}{r(\varepsilon)} = 0$ the latter expression behaves like $2\tilde\eta^2 K \frac{r(\varepsilon)}{\varepsilon}$ as $\varepsilon \to 0$. Putting these estimates together yields the claimed asymptotic bound with $\tilde\eta$ instead of η. Finally, letting $\tilde\eta \to \eta$ establishes (2.5).

The uniformity of (2.5) in the class of all diffusions that satisfy (2.3) is seen as follows. The crucial inequality (2.9) contains a bound which is independent of X^ε, since ϕ is defined by means of h, ε, $\tilde\eta$ and R_0 only, i.e. ε_1 and R_1 are independent of X^ε. □

As an immediate consequence, we obtain the following result by setting $r(\varepsilon) = T$ in Theorem 2.3.

2.5 Corollary. *For all $R \geq R_1$ and $T > 0$ we have*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left[\tau_R^\varepsilon \leq T\right] \leq -\eta R.$$

2.2 Extensions of Freidlin-Wentzell estimates

The classical large deviations theory by Freidlin and Wentzell describes the small noise asymptotics of the time homogeneous diffusion (1.6) under global Lipschitz assumptions on the drift b and the diffusion coefficient σ. For a Brownian particle that travels in a potential landscape this assumption is typically not fulfilled. In this case b is the spatial gradient of the potential function, hence it satisfies dissipativity properties such as the ones introduced in the previous section. Moreover, in the setting with metastable states of Part II of this thesis a weak time inhomogeneity as well as ε-dependent drift terms have to be considered.

In this section we shall present an account of the Freidlin-Wentzell theory that takes care of some of the required generalizations, i.e. we allow the drift and diffusion coefficients to be locally Lipschitz and ε-dependent. Our presentation follows the approach given by Azencott [2] and extensions thereof by Priouret [39] and Baldi [3].

2.2.1 The large deviations principle

We shall prove a large deviations principle for the family $(X^\varepsilon)_{\varepsilon>0}$ of diffusions governed by the d-dimensional stochastic differential equation

$$dX_t^\varepsilon = b^\varepsilon(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\sigma^\varepsilon(X_t^\varepsilon)\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d \tag{2.10}$$

on a finite time interval $[0, T]$ for a fixed time horizon $T > 0$. We suppose that we are given a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, so that W is a Brownian motion under $\mathbb{P}$ and that, for each fixed $\varepsilon > 0$, X^ε is a strong solution w.r.t. this Brownian motion that exists at least up to time T. This may be guaranteed e.g. by the assumption that b^ε and σ^ε satisfy the conditions discussed in the previous section.

Throughout this section we shall make the following assumptions. Recall that H_{0T}^x denotes the Cameron-Martin space of absolutely continuous functions with square integrable derivative starting at $x \in \mathbb{R}^d$. We abbreviate $H_{0T} = H_{0T}^0$.

2.6 Assumption.

(i) b^ε and σ^ε are locally Lipschitz continuous for every $\varepsilon > 0$.

(ii) *There exist locally Lipschitz functions* $b : \mathbb{R}^d \to \mathbb{R}^d$ *and* $\sigma : \mathbb{R}^d \to \mathbb{R}^{d\times d}$ *such that*

$$\lim_{\varepsilon\to 0} b^\varepsilon(x) = b(x) \quad \text{and} \quad \lim_{\varepsilon\to 0} \sigma^\varepsilon(x) = \sigma(x)$$

for $x \in \mathbb{R}^d$, *uniformly on compact sets.*

(iii) *For all* $f \in H_{0T}$ *and* $x \in \mathbb{R}^d$ *the equation*

$$\dot{g}_t = b(g_t) + \sigma(g_t)\dot{f}_t, \quad g_0 = x$$

has a solution on $[0,T]$. *We write* $g = S^x(f)$.

The proof of the LDP relies on the following main estimate, which supports the intuitive idea that the contraction principle is formally applicable. According to the latter, if $\sqrt{\varepsilon}W$ is close to $f \in H_{0T}$, then X^ε should be close to $g = S^{x_0}(f)$. The next theorem shows that this is indeed true, and that this closeness holds on exponential scales of any order. In the sequel, we write X^{ε,x_0} if we want to emphasize the initial condition of X^ε.

2.7 Theorem (Main estimate)**.** *For each* $R > 0$, $\delta > 0$ *and* $\alpha > 0$ *there exist* $\gamma > 0$, $\varrho > 0$ *and* $\varepsilon_0 > 0$ *such that*

$$\varepsilon \log \mathbb{P}\left[\rho_{0T}\Big(X^{\varepsilon,x}, S^{x_0}(f)\Big) > \delta,\ \rho_{0T}\Big(\sqrt{\varepsilon}\, W, f\Big) \le \gamma\right] \le -R$$

holds true for all $f \in H_{0T}$ *with* $J_{0T}(f) \le \alpha$, $x \in B_\varrho(x_0)$ *and* $\varepsilon \le \varepsilon_0$.

Before addressing the proof of Theorem 2.7 in subsequent sections, we demonstrate how it implies the LDP for the family (X^ε). This is the announced generalization of the classical Theorem 1.5. Recall that J_{0T} denotes the rate function of Brownian motion, from which the rate function of (X^ε) is derived by setting

$$I_{0T}^{x_0}(\varphi) = \inf\Big\{J_{0T}(f) :\ f \in H_{0T}, S^{x_0}(f) = \varphi\Big\}, \tag{2.11}$$

and denote by

$$\mathcal{I}_{0T}^{x_0}(A) = \inf_{\varphi\in A} I_{0T}^{x_0}(\varphi), \qquad A \subset C_{0T}, \tag{2.12}$$

the 'Cramer functional' of (X^ε). We summarize some of its properties (see Azencott [2], Proposition III.2.10, and Priouret [39], Proposition 6).

2.8 Proposition.

a) *For all* $\alpha > 0$ *and each compact* $L \subset \mathbb{R}^d$, *the level set*

$$\Big\{\varphi \in C_{0T} :\ I_{0T}^{\varphi_0}(\varphi) \le \alpha,\ \varphi_0 \in L\Big\}$$

is compact. Here φ_0 *denotes the starting point of* $\varphi \in C_{0T}$. *In particular, we have that* $I_{0T}^{x_0}$ *is a good rate function for each* $x_0 \in \mathbb{R}^d$.

b) If $I_{0T}^{x_0}(\varphi) < \infty$, then there exists $f \in H_{0T}$ such that $S^{x_0}(f) = \varphi$ and $J_{0T}(f) = I_{0T}^{x_0}(\varphi)$.

The large deviations principle in the form stated in the following theorem provides uniform bounds w.r.t. small neighborhoods of x_0.

2.9 Theorem (Large deviations principle)**.** *Under Assumption 2.6 the diffusions $(X^\varepsilon)_{\varepsilon>0}$ defined by* (2.10) *satisfy a large deviations principle in C_{0T} with good rate function $I_{0T}^{x_0}$. More precisely, for closed resp. open sets $F \subset C_{0T}$ and $G \subset C_{0T}$ we have*

$$\limsup_{\varepsilon\to 0,\ \varrho\to 0} \varepsilon \log \sup_{x\in B_\varrho(x_0)} \mathbb{P}_x \Big[X^\varepsilon \in F \Big] \le -\mathcal{I}_{0T}^{x_0}(F), \tag{2.13}$$

$$\liminf_{\varepsilon\to 0,\ \varrho\to 0} \varepsilon \log \inf_{x\in B_\varrho(x_0)} \mathbb{P}_x \Big[X^\varepsilon \in G \Big] \ge -\mathcal{I}_{0T}^{x_0}(G). \tag{2.14}$$

Here $\mathbb{P}_x$ indicates the initial condition of the diffusion.

Proof. In order to prove the lower bound, let $G \subset C_{0T}$ be an open set, let $\eta > 0$, and choose $\varphi \in G$ such that $I_{0T}^{x_0}(\varphi) \le \mathcal{I}_{0T}^{x_0}(G) + \eta$. Furthermore, let $f \in H_{0T}$ such that $S^{x_0}(f) = \varphi$ and $J_{0T}(f) = I_{0T}^{x_0}(\varphi)$. Let $\delta > 0$ such that $B_\delta(\varphi) \subset G$. Then for each $\gamma > 0$, $x \in \mathbb{R}^d$

$$\begin{aligned} \mathbb{P}\Big[X^{\varepsilon,x} \in G\Big] &\ge \mathbb{P}\Big[\rho_{0T}(X^{\varepsilon,x},\varphi) \le \delta\Big] \\ &\ge \mathbb{P}\Big[\rho_{0T}(\sqrt{\varepsilon}W, f) \le \gamma\Big] - \mathbb{P}\Big[\rho_{0T}(X^{\varepsilon,x},\varphi) > \delta, \rho_{0T}(\sqrt{\varepsilon}W, f) \le \gamma\Big]. \end{aligned} \tag{2.15}$$

By Schilder's theorem, the first probability on the r.h.s. is bounded from below

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\Big[\rho_{0T}(\sqrt{\varepsilon}W, f) \le \gamma\Big] \ \ge\ -J_{0T}(f) \ =\ -I_{0T}^{x_0}(\varphi) \ \ge\ -\mathcal{I}_{0T}^{x_0}(G) - \eta.$$

The second probability in (2.15) becomes arbitrarily small by Theorem 2.7: fix $\alpha \ge J_{0T}(f)$ and $R > \mathcal{I}_{0T}^{x_0}(G) + \eta$. Then we may find $\gamma > 0$, $\varrho > 0$ and $\varepsilon_0 > 0$ such that

$$\varepsilon \log \mathbb{P}\Big[\rho_{0T}(X^{\varepsilon,x},\varphi) > \delta,\ \rho_{0T}(\sqrt{\varepsilon}W, f) \le \gamma\Big] \le -R, \quad x \in B_\varrho(x_0),\ \varepsilon \le \varepsilon_0,$$

which implies

$$\liminf_{\varepsilon\to 0,\ \varrho\to 0} \varepsilon \log \inf_{x\in B_\varrho(x_0)} \mathbb{P}\Big[X^{\varepsilon,x} \in G\Big] \ge \min\{-R, -\mathcal{I}_{0T}^{x_0}(G) - \eta\} = -\mathcal{I}_{0T}^{x_0}(G) - \eta,$$

i.e. the lower bound is proved.

For the upper bound, let F be a closed set, let $0 < \alpha < \mathcal{I}_{0T}^{x_0}(F)$, and fix $R > \alpha$. Let $\varphi \in H_{0T}^{x_0}$ with $I_{0T}^{x_0}(\varphi) \le \alpha$. Then we may find $\delta = \delta_\varphi > 0$ such that $B_\delta(\varphi) \cap F = \emptyset$

and $f = f_\varphi \in \{J_{0T} \le \alpha\}$ such that $S^{x_0}(f) = \varphi$. By Theorem 2.7 there exist $\gamma = \gamma_\varphi$, $\varrho = \varrho_\varphi$ and $\varepsilon_\varphi > 0$ such that

$$\varepsilon \log \mathbb{P}\left[\rho_{0T}(X^{\varepsilon,x},\varphi) > \delta,\ \rho_{0T}(\sqrt{\varepsilon}W, f) \le \gamma\right] \le -R, \quad x \in B_\varrho(x_0),\ \varepsilon \le \varepsilon_\varphi. \qquad (2.16)$$

Since the open neighborhoods $\{B_{\gamma_\varphi}(f_\varphi) :\ \varphi \in H^{x_0}_{0T}, I^{x_0}_{0T}(\varphi) \le \alpha\}$ cover the compact $\{J_{0T} \le \alpha\}$, we may extract a finite sub-cover $\{B_{\gamma_i}(f_i) :\ i = 1, \ldots, k\}$. Set $A = \cup_{i=1}^k B_{\gamma_i}(f_i)$ and $\varphi_i = S^{x_0}(f_i)$. To each φ_i ($i \in \{1, \ldots, k\}$) corresponds a choice of $\delta_i > 0$, $\varrho_i > 0$ and $\varepsilon_i > 0$ such that (2.16) holds true for φ_i. Let $\varepsilon_0 = \min\{\varepsilon_1, \ldots, \varepsilon_k\}$ and $\varrho = \min\{\varrho_1, \ldots, \varrho_k\}$. Recalling $B_{\delta_i}(\varphi_i) \cap F = \emptyset$ for $i = 1, \ldots, k$, we see that for $\varepsilon \le \varepsilon_0$ and $x \in B_\varrho(x_0)$

$$\begin{aligned}
\mathbb{P}\left[X^{\varepsilon,x} \in F\right] &\le \mathbb{P}\left[X^{\varepsilon,x} \in F,\ \sqrt{\varepsilon}W \in A\right] + \mathbb{P}\left[\sqrt{\varepsilon}W \in A^c\right] \\
&\le \mathbb{P}\left[\sqrt{\varepsilon}W \in A^c\right] + \sum_{i=1}^k \mathbb{P}\left[\rho_{0T}(X^{\varepsilon,x},\varphi_i) > \delta_i,\ \rho_{0T}(\sqrt{\varepsilon}W, f_i) \le \gamma_i\right] \\
&\le e^{-\alpha/\varepsilon} + k e^{-R/\varepsilon}.
\end{aligned}$$

Since $R > \alpha$ this permits us to conclude. $\square$

The uniformity of the LDP w.r.t. small neighborhoods of x_0 means in particular that the LDP still holds true for a family of diffusions with ε-dependent initial conditions that converge to x_0 as $\varepsilon \to 0$. Moreover, it always implies a uniform large deviations bound on compact subsets of $\mathbb{R}^d$ as follows.

2.10 Corollary. *Let $L \subset \mathbb{R}^d$ be a compact set. For closed resp. open sets $F \subset C_{0T}$ and $G \subset C_{0T}$ we have*

$$\begin{aligned}
\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x_0 \in L} \mathbb{P}_{x_0}\left[X^\varepsilon \in F\right] &\le -\inf_{x_0 \in L} \mathcal{I}^{x_0}_{0T}(F), \\
\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{x_0 \in L} \mathbb{P}_{x_0}\left[X^\varepsilon \in G\right] &\ge -\sup_{x_0 \in L} \mathcal{I}^{x_0}_{0T}(G).
\end{aligned}$$

The proof of this result is exactly the same as the one of Corollary 5.6.15 in [17].

2.2.2 A special case of the main estimate

The main estimate of Theorem 2.7 provides an exponential bound for the probability that the diffusion X^ε is *not* in the delta tube of $S^{x_0}(f)$, while $\sqrt{\varepsilon}W$ is close to $f \in H_{0T}$ at the same time. Of course the latter is itself a rare event by Schilder's theorem.
The idea to prove this main estimate is to perform a Girsanov type change of measure, so that under the new measure Q^ε we have $\sqrt{\varepsilon}W - f = \sqrt{\varepsilon}B^\varepsilon$ with a Brownian motion B^ε. This means that f is close to $\sqrt{\varepsilon}W$ under Q^ε, and it shall turn out that

the main estimate reduces to the special case $f = 0$ under Q^ε, but with the new non-autonomous drift $b^\varepsilon(x) + \sigma^\varepsilon(x)\dot{f}_t$.

This reasoning motivates us to deal with the following situation in this subsection. In a slightly more general setting, we allow for time-dependent drift terms, and consider the family $(Y^\varepsilon)_{\varepsilon>0}$ of solutions to the stochastic differential equation

$$dY_t^\varepsilon = c^\varepsilon(t, Y_t^\varepsilon)\, dt + \sqrt{\varepsilon}\sigma^\varepsilon(Y_t^\varepsilon)\, dW_t, \qquad Y_0^\varepsilon = y_0 \in \mathbb{R}^d\,. \tag{2.17}$$

on the time interval $[0, T]$. Throughout, we shall make the following assumptions for the coefficients of Y^ε.

2.11 Assumption.

(i) *The vector fields* $c^\varepsilon : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *converge to some vector field* $c : [0, T] \times \mathbb{R}^d \to \mathbb{R}^d$ *in the sense that*

$$\lim_{\varepsilon\to 0} \int_0^T \sup_{y\in\mathbb{R}^d} \Big\| c^\varepsilon(t, y) - c(t, y) \Big\| dt = 0.$$

(ii) *The matrix fields* σ^ε *converge uniformly on* $\mathbb{R}^d$ *to some matrix field* $\sigma : \mathbb{R}^d \to \mathbb{R}^d$ *which is bounded and globally Lipschitz.*

(iii) *There exists a function* $\chi \in L^2([0, T])$ *such that*

$$\|c^\varepsilon(t, y)\| + \|c(t, y)\| \le \chi(t) \quad \textit{for } 0 \le t \le T,\ y \in \mathbb{R}^d\,.$$

(iv) *There exists a function* $\kappa \in L^1([0, T])$ *such that*

$$\|c(t, y_1) - c(t, y_2)\| \le \kappa(t)\|y_1 - y_2\| \quad \textit{for } 0 \le t \le T,\ y_1, y_2 \in \mathbb{R}^d\,.$$

A first step consists of a discretization that approximates the diffusions Y^ε well enough in the sense of large deviations. For $n \in \mathbb{N}$ let $t_k := t_k^n := \frac{kT}{n}$, and let

$$Y_t^{\varepsilon,n} := Y_{t_k}^\varepsilon \qquad \text{for } t_k \le t < t_{k+1}.$$

We denote by $\|\cdot\|_{0T}$ the sup norm that corresponds to ρ_{0T}, but we employ this norm also for discontinuous functions.

2.12 Lemma. *For any* $\delta > 0$ *we have*

$$\lim_{n\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left[\|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} > \delta\right] = -\infty,$$

uniformly w.r.t. the initial condition $y_0 \in \mathbb{R}^d$.

Proof. We have

$$\mathbb{P}\Big[\|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} > \delta\Big] = \mathbb{P}\Big[\bigcup_{k=0}^{n-1}\Big\{\sup_{t_k \le t < t_{k+1}} \|Y_t^\varepsilon - Y_t^{\varepsilon,n}\| > \delta\Big\}\Big].$$

Now for $t_k \le t < t_{k+1}$

$$Y_t^\varepsilon - Y_t^{\varepsilon,n} = \int_{t_k}^t c^\varepsilon(s, Y_s^\varepsilon)ds + \sqrt{\varepsilon}\int_{t_k}^t \sigma^\varepsilon(Y_s^\varepsilon)dW_s,$$

and therefore

$$\begin{aligned}\mathbb{P}\Big[\|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} > \delta\Big] \;&\le\; \sum_{k=0}^{n-1} \;\mathbb{P}\Big[\sup_{t_k \le t < t_{k+1}} \Big\|\int_{t_k}^t c^\varepsilon(s, Y_s^\varepsilon)ds\Big\| > \frac{\delta}{2}\Big] \\ &\quad + \;\mathbb{P}\Big[\sup_{t_k \le t < t_{k+1}} \Big\|\sqrt{\varepsilon}\int_{t_k}^t \sigma^\varepsilon(Y_s^\varepsilon)dW_s\Big\| > \frac{\delta}{2}\Big]. \end{aligned} \qquad (2.18)$$

The first term on the r.h.s. of this inequality vanishes if n is sufficiently large. Indeed, by the integrability assumption for c^ε we have for $t_k \le t < t_{k+1}$

$$\Big|\int_{t_k}^t c^\varepsilon(s, Y_s^\varepsilon)ds\Big| \le \int_{t_k}^t \chi(s)ds \le \sqrt{\frac{T}{n}}\,\|\chi\|_2\,,$$

which becomes smaller than $\delta/2$ for large n. The second term in (2.18) can be bounded as follows. Use Assumption 2.11 to find $\varepsilon_0 > 0$ and $m > 0$ such that $\|\sigma^\varepsilon(y)\| \le m$ for $y \in \mathbb{R}^d$, $\varepsilon \le \varepsilon_0$. Then the exponential martingale inequality (Proposition A.1) yields

$$\mathbb{P}\Big[\sup_{t_k \le t < t_{k+1}} \Big\|\sqrt{\varepsilon}\int_{t_k}^t \sigma^\varepsilon(Y_s^\varepsilon)dW_s\Big\| > \frac{\delta}{2}\Big] \;\le\; 2d\exp\Big(-\frac{n\delta^2}{8Tdm^2\varepsilon}\Big), \qquad \varepsilon \le \varepsilon_0.$$

We conclude that there exists $n_0 \in \mathbb{N}$ such that for $n \ge n_0$ and $\varepsilon \le \varepsilon_0$

$$\mathbb{P}\Big[\|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} > \delta\Big] \;\le\; 2dn\exp\Big(-\frac{n\delta^2}{8Tdm^2\varepsilon}\Big).$$

Sending $\varepsilon \to 0$ and $n \to \infty$ finishes the argument. The uniformity w.r.t. y_0 is obvious. □

In the next lemma we show that the exponential approximation of Y^ε results in an exponential approximation of the martingale part of Y^ε.

2.13 Lemma. *Let*

$$M_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(Y_s^\varepsilon)dW_s \qquad \textit{and} \qquad M_t^{\varepsilon,n} = \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(Y_s^{\varepsilon,n})dW_s.$$

For all $\delta > 0$

$$\lim_{\beta\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\Big[\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta,\; \|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} \le \beta\Big] = -\infty,$$

uniformly w.r.t. $n \in \mathbb{N}$, $y_0 \in \mathbb{R}^d$.

Proof. Fix $\delta > 0$ and $\beta > 0$. We shall separately estimate probabilities according to the decomposition

$$\begin{aligned}&\Big\{\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta\Big\} \qquad (2.19)\\ &\subset \Big\{\|N^\varepsilon\|_{0T} > \tfrac{\delta}{3}\Big\} \cup \Big\{\|N^{\varepsilon,n}\|_{0T} > \tfrac{\delta}{3}\Big\} \cup \Big\{\Big\|\sqrt{\varepsilon}\int_0^\cdot \sigma(Y_s^\varepsilon) - \sigma(Y_s^{\varepsilon,n})dW_s\Big\|_{0T} > \tfrac{\delta}{3}\Big\}.\end{aligned}$$

By uniform convergence, we may find $\varepsilon_0 > 0$ such that $\sup_{y\in\mathbb{R}^d} \|\sigma^\varepsilon(y) - \sigma(y)\| \le \beta$ for $\varepsilon \le \varepsilon_0$. Thus,

$$N_t^\varepsilon = \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(Y_s^\varepsilon) - \sigma(Y_s^\varepsilon)dW_s \quad \text{and} \quad N_t^{\varepsilon,n} = \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(Y_s^{\varepsilon,n}) - \sigma(Y_s^{\varepsilon,n})dW_s$$

are martingales, and the exponential inequality (Proposition A.1) yields

$$\mathbb{P}\Big[\|N^\varepsilon\|_{0T} > \delta\Big] \le 2d\exp\Big(-\frac{\delta^2}{2\varepsilon T d\,\beta^2}\Big), \quad \varepsilon \le \varepsilon_0,$$

and the same bound holds true for $N^{\varepsilon,n}$. In order to estimate the probability of the third set in (2.19), let $\tau := \inf\{t > 0 : \ \|Y_t^\varepsilon - Y_t^{\varepsilon,n}\| > \beta\}$, and let K denote a Lipschitz constant for σ. By the exponential inequality we have

$$\begin{aligned}&\mathbb{P}\Big[\Big\{\Big\|\sqrt{\varepsilon}\int_0^\cdot \sigma(Y_s^\varepsilon) - \sigma(Y_s^{\varepsilon,n})dW_s\Big\|_{0T} > \delta\Big\} \cap \Big\{\Big\|Y^\varepsilon - Y^{\varepsilon,n}\Big\|_{0T} \le \beta\Big\}\Big]\\ &= \mathbb{P}\Big[\Big\{\Big\|\sqrt{\varepsilon}\int_0^\cdot \mathbf{1}_{\{\tau>s\}}\sigma(Y_s^\varepsilon) - \sigma(Y_s^{\varepsilon,n})dW_s\Big\|_{0T} > \delta\Big\} \cap \Big\{\tau > T\Big\}\Big]\\ &\le 2d\exp\Big(-\frac{\delta^2}{2\varepsilon T d K^2\beta^2}\Big),\end{aligned}$$

since the integrand is bounded by βK. By combining these exponential bounds we obtain for all $n \in \mathbb{N}$

$$\begin{aligned}&\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\Big[\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta,\ \|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} \le \beta\Big]\\ &\qquad\le \max\Big\{-\frac{(\delta/3)^2}{2Td\,\beta^2}, -\frac{(\delta/3)^2}{2TdK^2\beta^2}\Big\} = -\frac{\delta^2}{18Td\,\beta^2}\min\Big\{1, K^{-2}\Big\}.\end{aligned}$$

This bound is independent of y_0, and it tends to $-\infty$ as $\beta \to 0$. $\square$

The next lemma provides a pre-stage of the main estimate for the martingale part of the diffusion.

2.14 Lemma. *For each $\delta > 0$*

$$\lim_{\gamma\to 0}\limsup_{\varepsilon\to 0}\varepsilon\log\mathbb{P}\Big[\|M^\varepsilon\|_{0T} > \delta,\ \|\sqrt{\varepsilon}\,W\|_{0T} \le \gamma\Big] = -\infty,$$

uniformly w.r.t. the initial condition y_0 of Y^ε.

Proof. Since $\{\|M^\varepsilon\|_{0T} > \delta\} \subset \{\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta/2\} \cup \{\|M^{\varepsilon,n}\|_{0T} > \delta/2\}$, we have for $\beta, \gamma > 0$

$$\Big\{\|M^\varepsilon\|_{0T} > \delta,\ \|\sqrt{\varepsilon}\,W\|_{0T} \leq\ \gamma\Big\}$$
$$\subset \Big\{\Big\|Y^\varepsilon - Y^{\varepsilon,n}\Big\|_{0T} > \beta\Big\} \cup \Big\{\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta/2,\ \Big\|Y^\varepsilon - Y^{\varepsilon,n}\Big\|_{0T} \leq \beta\Big\}$$
$$\cup \Big\{\|M^{\varepsilon,n}\|_{0T} > \delta/2,\ \Big\|Y^\varepsilon - Y^{\varepsilon,n}\Big\|_{0T} \leq \beta,\ \|\sqrt{\varepsilon}\,W\|_{0T} \leq \gamma\Big\}.$$

Now fix $R > 0$. By Lemmas 2.13 and 2.12 we find $\beta_0 > 0$, $\varepsilon_0 > 0$ and $n_0 = n_0(\beta_0) \in \mathbb{N}$ such that

$$\mathbb{P}\Big[\|M^\varepsilon - M^{\varepsilon,n}\|_{0T} > \delta/2,\ \|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} \leq \beta_0\Big] \leq e^{-R/\varepsilon}$$

and

$$\mathbb{P}\Big[\|Y^\varepsilon - Y^{\varepsilon,n}\|_{0T} > \beta_0\Big] \leq e^{-R/\varepsilon}$$

hold true for $n \geq n_0$ and $\varepsilon \leq \varepsilon_0$. Furthermore, by definition of $M^{\varepsilon,n}$

$$M_t^{\varepsilon,n} = \sqrt{\varepsilon}\int_0^t \sigma^\varepsilon(Y_s^{\varepsilon,n})dW_s = \sqrt{\varepsilon}\sum_{k=0}^{n-1}\sigma^\varepsilon\Big(Y_{t_k}^\varepsilon\Big)\Big(W_{t_{k+1}\wedge t} - W_{t_k\wedge t}\Big).$$

Using the uniform convergence $\sigma^\varepsilon \to \sigma$, we find $m > 0$, $\varepsilon_1 > 0$ such that

$$\|M^{\varepsilon,n}\|_{0T} \leq 2mn\gamma \qquad \text{on} \quad \Big\{\|\sqrt{\varepsilon}\,W\|_{0T} \leq \gamma\Big\}$$

for $\varepsilon \leq \varepsilon_1$. We deduce that

$$\Big\{\|M^{\varepsilon,n_0}\|_{0T} > \delta/2,\ \|\sqrt{\varepsilon}\,W\|_{0T} \leq \gamma\Big\} = \emptyset$$

for $\gamma \leq \gamma_0 := \frac{\delta}{4mn_0}$, which finally leads to

$$\mathbb{P}\Big[\|M^\varepsilon\|_{0T} > \delta,\ \|\sqrt{\varepsilon}\,W\|_{0T} \leq \gamma\Big] \leq e^{-R/\varepsilon}$$

for $\varepsilon \leq \varepsilon_0 \wedge \varepsilon_1$, $\gamma \leq \gamma_0$. The uniformity w.r.t. y_0 is again obvious. □

In the next proposition, the estimate for M^ε is carried over to Y^ε. Recall that χ and κ are the functions introduced in Assumption 2.11 that control the growth of c and c^ε.

2.15 Proposition. *Let ζ denote the solution of the ODE*

$$\dot{\zeta}_t = c(t, \zeta_t), \quad \zeta_0 = y.$$

For $R, \delta > 0$ there exist $\gamma > 0$, $\varrho > 0$ and $\varepsilon_0 > 0$ such that

$$\mathbb{P}\Big[\Big\{\rho_{0T}(Y^\varepsilon, \zeta) > \delta\Big\} \cap \Big\{\Big\|\sqrt{\varepsilon}W\Big\|_{0T} \leq \gamma\Big\}\Big]\ \leq\ \exp\Big\{-\tfrac{R}{\varepsilon}\Big\}$$

for all $y_0 \in \mathbb{R}^d$, $y \in B_\varrho(y_0)$ and $\varepsilon \leq \varepsilon_0$.

Proof. Let $a \geq \|\kappa\|_1$, and fix $\delta > 0$. Choose $\varepsilon_1 > 0$ according to Assumption 2.11 (i) such that

$$\int_0^T \sup_{y \in \mathbb{R}^d} \Big\| c^\varepsilon(t,y) - c(t,y) \Big\| dt \;\leq\; \frac{\delta}{4} e^{-aT}$$

for $\varepsilon \leq \varepsilon_1$. Hence, for $0 \leq t \leq T$ and $\varepsilon \leq \varepsilon_1$ we have

$$\begin{aligned}
\Big\| Y_t^\varepsilon - \zeta_t \Big\| &\leq \|y_0 - y\| + \Big\| M_t^\varepsilon + \int_0^t c^\varepsilon(s, Y_s^\varepsilon) - c(s,\zeta_s) ds \Big\| \\
&\leq \|y_0 - y\| + \Big\| M_t^\varepsilon \Big\| + \int_0^t \Big\| c^\varepsilon(s, Y_s^\varepsilon) - c(s, Y_s^\varepsilon) \Big\| ds \\
&\qquad + \int_0^t \Big\| c(s, Y_s^\varepsilon) - c(s,\zeta_s) \Big\| ds \\
&\leq \|y_0 - y\| + \Big\| M_t^\varepsilon \Big\| + \frac{\delta}{4} e^{-aT} + \int_0^t \kappa(s) \Big\| Y_s^\varepsilon - \zeta_s \Big\| ds.
\end{aligned}$$

Thus, if $\|y_0 - y\| \leq \frac{\delta}{4} e^{-aT} =: \varrho$, we may invoke Gronwall's lemma to deduce that

$$\|Y^\varepsilon - \zeta\|_{0T} \;\leq\; \Big(\|y_0 - y\| + \|M^\varepsilon\|_{0T} + \frac{\delta}{4} e^{-aT} \Big) \exp\Big\{ \int_0^T \kappa(s) ds \Big\} \;\leq\; \frac{\delta}{2} + \|M^\varepsilon\|_{0T}\, e^{aT}.$$

This yields

$$\mathbb{P}\Big[\Big\{ \rho_{0T}(Y^\varepsilon, \zeta) > \delta \Big\} \cap \Big\{ \Big\| \sqrt{\varepsilon} W \Big\| \leq \gamma \Big\} \Big] \leq \mathbb{P}\Big[\|M^\varepsilon\|_{0T}\, e^{aT} > \frac{\delta}{2}, \Big\| \sqrt{\varepsilon} W \Big\|_{0T} \leq \gamma \Big],$$

and an appeal to Lemma 2.14 finishes the proof. □

2.2.3 Proof of the main estimate

We are now prepared to prove the main estimate. Using Girsanov's theorem, we transform the general case of Theorem 2.7 to the special case treated in the previous subsection.

Proof of Theorem 2.7. Fix $\alpha > 0$, and let $f \in \{J_{0T} \leq \alpha\}$. Define the exponential density

$$Z_T^\varepsilon := \exp\Big\{ \frac{1}{\sqrt{\varepsilon}} \int_0^T \dot{f}_s dW_s - \frac{1}{2\varepsilon} \int_0^T \|\dot{f}_s\|^2 ds \Big\},$$

and let Q^ε be the probability measure defined by $\frac{dQ^\varepsilon}{d\mathbb{P}} = Z_T^\varepsilon$. By Girsanov's theorem,

$$B_t^\varepsilon = W_t - \frac{1}{\sqrt{\varepsilon}} f_t, \quad 0 \leq t \leq T,$$

is a Brownian motion under Q^ε, and $dW_t = dB_t^\varepsilon + \frac{1}{\sqrt{\varepsilon}}\dot{f}_t dt$. Therefore, under Q^ε we have

$$\begin{aligned} dX_t^\varepsilon &= \Big[b^\varepsilon(X_t^\varepsilon) + \sigma^\varepsilon(X_t^\varepsilon)\dot{f}_t\Big]dt + \sqrt{\varepsilon}\sigma^\varepsilon(X_t^\varepsilon)dB_t^\varepsilon \\ &= c^\varepsilon(t, X_t^\varepsilon)dt + \sqrt{\varepsilon}\sigma^\varepsilon(X_t^\varepsilon)dB_t^\varepsilon, \end{aligned}$$

where $c^\varepsilon(t,x) = b^\varepsilon(x) + \sigma^\varepsilon(x)\dot{f}_t$. The limit of this drift as $\varepsilon \to 0$ is given by

$$c(t,x) := b(x) + \sigma(x)\dot{f}_t.$$

In order to prove the claimed asymptotics, we proceed in two steps. We will first assume that the coefficients are bounded, so that we are in the setting of Assumption 2.11. In a second step we relax these assumptions via a localization argument.

Step 1: Assume that b and σ are bounded and globally Lipschitz with constants m and K, respectively, and suppose that the convergence $b^\varepsilon \to b$ and $\sigma^\varepsilon \to \sigma$ is uniform on $\mathbb{R}^d$. Then we have

$$\begin{aligned} \|c^\varepsilon(t,x)\| + \|c(t,x)\| &\leq 2m(1 + \|\dot{f}_t\|), \\ \|c(t,x) - c(t,y)\| &\leq K(1 + \|\dot{f}_t\|)\,\|x - y\|, \end{aligned}$$

and

$$\sup_{x\in\mathbb{R}^d} \|c^\varepsilon(t,x) - c(t,x)\| \leq (1 + \|\dot{f}_t\|) \sup_{x\in\mathbb{R}^d} \Big(\|b^\varepsilon(x) - b(x)\| + \|\sigma^\varepsilon(x) - \sigma(x)\|\Big),$$

i.e. Assumption 2.11 is satisfied with $\chi(t) = 2m(1 + \|\dot{f}_t\|)$ and $\kappa(t) = K(1 + \|\dot{f}_t\|)$. Fix $R > 0$ and $\delta > 0$, and let

$$\begin{aligned} A^{\varepsilon,x} &:= \Big\{\rho_{0T}(X^{\varepsilon,x}, S^{x_0}(f)) > \delta,\ \rho_{0T}(\sqrt{\varepsilon}\,W, f) \leq \gamma\Big\} \\ &= \Big\{\rho_{0T}(X^{\varepsilon,x}, S^{x_0}(f)) > \delta,\ \Big\|\sqrt{\varepsilon}B^\varepsilon\Big\|_{0T} \leq \gamma\Big\} \end{aligned}$$

denote the set of interest. By Proposition 2.15 we may find $\varrho > 0$, $\gamma > 0$ and $\varepsilon_0 > 0$ such that

$$Q^\varepsilon[A^{\varepsilon,x}] \leq \exp\Big\{-\tfrac{2(R+\alpha)}{\varepsilon}\Big\}, \qquad \varepsilon \leq \varepsilon_0,\ x \in B_\varrho(x_0).$$

Moreover, since

$$\frac{d\,\mathbb{P}}{dQ^\varepsilon} = \frac{1}{Z_T} = \exp\Big\{-\frac{1}{\sqrt{\varepsilon}}\int_0^T \dot{f}_s dB_s^\varepsilon - \frac{1}{2\varepsilon}\int_0^T \|\dot{f}_s\|^2 ds\Big\},$$

Schwarz's inequality yields

$$\mathbb{P}[A^{\varepsilon,x}] = \int_{A^{\varepsilon,x}} \frac{1}{Z_T}\, dQ^\varepsilon \leq \Big(Q^\varepsilon[A^{\varepsilon,x}]\,\mathbb{E}_{Q^\varepsilon}\Big[Z_T^{-2}\Big]\Big)^{1/2},$$

where

$$\mathbb{E}_{Q^\varepsilon}\Big[Z_T^{-2}\Big] = \mathbb{E}_{Q^\varepsilon}\Big[\exp\Big\{-\frac{2}{\sqrt{\varepsilon}}\int_0^T \dot{f}_s\,dB_s^\varepsilon - \frac{1}{\varepsilon}\int_0^T \|\dot{f}_s\|^2 ds\Big\}\Big] = \exp\big\{\tfrac{2}{\varepsilon}\,J_{0T}(f)\big\}.$$

Therefore

$$\mathbb{P}[A^{\varepsilon,x}] \leq \exp\big\{-\tfrac{R+\alpha}{\varepsilon}\big\}\exp\big\{\tfrac{1}{\varepsilon}\,J_{0T}(f)\big\} \leq \exp\big\{-\tfrac{R}{\varepsilon}\},\qquad \varepsilon\leq\varepsilon_0,\ x\in B_\varrho(x_0),$$

which is the desired exponential bound.

Step 2: In order to treat the general case, we cut the coeffcients and construct new drift and diffusion coeffcients that are globally Lipschitz and bounded. Let $r > 0$ such that the δ-tube around $S^{x_0}(f)$ is contained in $B_r(0)$. For $x \in \mathbb{R}^d$ let

$$\tilde{b}^\varepsilon(x) = \begin{cases} b^\varepsilon(x), & \text{if } \|x\| \leq r,\\ b^\varepsilon\big(\frac{x}{\|x\|}r\big), & \text{if } \|x\| > r,\end{cases}$$

and define $\tilde{b}$ and $\tilde{\sigma}^\varepsilon$, $\tilde{\sigma}$ analoguously. These functions are bounded and globally Lipschitz, and they evidently converge uniformly on $\mathbb{R}^d$ as $\varepsilon \to 0$. Furthermore, if $\tilde{X}^\varepsilon$ denotes the solution of the SDE (2.10) with the new coeffcients, we have $\tilde{X}^\varepsilon = X^\varepsilon$ before the exit from $B_r(0)$. Hence for all $x \in B_\varrho(x_0)$

$$\begin{aligned}&\mathbb{P}\Big[\rho_{0T}(X^{\varepsilon,x}, S^{x_0}(f)) > \delta,\ \rho_{0T}(\sqrt{\varepsilon}\,W, f) \leq \gamma\Big]\\ &\qquad = \mathbb{P}\Big[\rho_{0T}(\tilde{X}^{\varepsilon,x}, S^{x_0}(f)) > \delta,\ \rho_{0T}(\sqrt{\varepsilon}\,W, f) \leq \gamma\Big].\end{aligned}$$

This is exponentially small by the first step. $\square$

Chapter 3

The exit problem: Kramers' law

The main results obtained in this thesis are motivated by the well known Kramers' law, which was given a mathematically precise meaning by Freidlin and Wentzell [19]. This law provides a desription of the small noise asymptotics of the first exit time of a time homogeneous diffusion from a bounded 'metastable' set. In this chapter, we briefly recall this exit law and introduce the concept of quasi-potentials. A detailed presentation may be found in section 5.7 of [17].

For simplicity, we restrict ourselves to the case of additive Brownian noise, and consider the family of solutions of the autonomous d-dimensional stochastic differential equation

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d\,. \tag{3.1}$$

Here W is a d-dimensional Brownian motion, and the vector field b is assumed to be globally Lipschitz. We are interested in the small noise behavior of the first exit time

$$\tau_D^\varepsilon := \inf\left\{t \geq 0 :\ X_t^\varepsilon \notin D\right\}$$

of X^ε from a bounded domain $D \subset \mathbb{R}^d$, i.e. an open, bounded and connected set. The large deviations principle (Theorem 2.9) states that the trajectories of the diffusion X^ε are attracted to the deterministic dynamical system

$$\dot{\xi} = b(\xi), \quad \xi_0 = x_0. \tag{3.2}$$

as noise tends to zero. The probabilities of X^ε deviating from ξ are exponentially small in ε, and the diffusion will certainly exit from D within a certain time interval if the deterministic path ξ exits. More precisely, the large deviations principle yields as an immediate consequence the asymptotics of the exit time distribution, namely

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_{x_0}(\tau_D^\varepsilon \leq t) = -\inf\{V(x_0, y, s) :\ 0 \leq s \leq t, y \notin D\}$$

for all $x_0 \in D$ and $t > 0$. Here $V(x, y, t)$ denotes the so-called *cost function*, which describes the asymptotic cost of the diffusion X^ε to go from x to y in state space within time t. For $t \geq 0$, $x, y \in \mathbb{R}^d$ it is given by

$$V(x, y, t) = \inf \left\{ I_{0t}^x(\varphi) : \varphi \in C_{0t}, \varphi_0 = x, \varphi_t = y \right\}, \tag{3.3}$$

where $I_{0t}^{x_0}$ is the rate function of X^ε given by (1.12). The corresponding *quasi-potential*

$$V(x, y) = \inf_{t>0} V(x, y, t) \tag{3.4}$$

describes the cost of going from x to y eventually.

The problem of diffusion exit involves an analysis for the rare event that the diffusion leaves the domain D although the deterministic path stays inside, i.e. it is concerned with an exit which is triggered by noise only. In this sense the domain D is a metastable set for the diffusion. More precisely, assume that the following conditions hold true:

(i) The system (3.2) possesses a unique stable equilibrium point x^* in D.
(ii) The solutions of (3.2) satisfy

$$\xi_0 \in D \quad \Longrightarrow \quad \xi_t \in D \ \ \forall t > 0 \quad \text{and} \quad \lim_{t \to \infty} \xi_t = x^*. \tag{3.5}$$

These conditions allow for characteristic boundaries, since $\lim_{t \to \infty} \xi_t = x^*$ is required only if $\xi_0 \in D$, but not in case $\xi_0 \in \partial D$. This means in particular that D may be chosen as the domain of attraction of a local minimum in the potential gradient case. We have the following result about the asymptotics of the exit time, see [19] and [17].

3.1 Theorem (Kramers' law). *Let $D \subset \mathbb{R}^d$ be a bounded domain that satisfies the above stated conditions (i) and (ii). Assume furthermore that*

$$V^* := \inf_{y \in \partial D} V(x^*, y) < \infty.$$

Then for all $x_0 \in D$

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_0} \left[\tau_D^\varepsilon \right] = V^*,$$

and

$$\lim_{\varepsilon \to 0} \mathbb{P}_{x_0} \left[e^{\frac{V^* - \delta}{\varepsilon}} < \tau_D^\varepsilon < e^{\frac{V^* + \delta}{\varepsilon}} \right] = 1 \quad \textit{for all } \delta > 0.$$

If b is derived from a potential U in $\mathbb{R}^d$, i.e. the deterministic system (3.2) is conservative, it is well known that V^* is twice the minimal difference ΔU of potential energy between x^* and ∂D. The term 'quasi-potential' originates in this link between U and V. It explains the physicists' intuitive formula $\tau_D^\varepsilon \approx \exp\left\{\frac{2\Delta U}{\varepsilon}\right\}$. The following lemma is a special case of [19], Theorem 4.3.1.

3.2 Lemma. *Assume that* $b = -\nabla U$ *for a smooth potential* $U : \mathbb{R}^d \to \mathbb{R}^d$*. Then*

$$V(x^*, y) = 2(U(y) - U(x^*)) \tag{3.6}$$

for all $y \in \overline{D}$ *such that* $U(y) \leq \min_{z \in \partial D} U(z)$*. In particular,* $V^* = 2 \inf_{y \in \partial D}(U(y) - U(x^*))$*.*

In the gradient case of Lemma 3.2 there exist finer results that go beyond the scope of large deviations theory, which provide sub-exponential pre-factors of the exit time, see Bovier at al. [12]. Moreover, Day ([15]) has shown that $\tau_D^\varepsilon / \mathbb{E}(\tau_D^\varepsilon)$ is asymptotically exponentially distributed with parameter one. For further classical results about the exit problem we refer to [19], [15], and [49]. A nice survey of the exit problem is given in [16].

Part II

Transition times and stochastic resonance

Chapter 4

Introduction

The physical phenomenon of stochastic resonance (SR) has been discovered around the beginning of the 1980s. Its investigation took its origin in a stochastic toy model from climatology ([36], [6]), to give a qualitative explanation for the almost periodic recurrence of cold and warm ages (glacial cycles) in paleoclimatic data. Since then, a lively field of research on the topic has evolved in the physical community. See [21] for an overview.

Roughly, the effect of SR may be described as follows:

> *Intrinsic or exterior periodicity properties of a system are exposed or amplified by the influence of noise of a properly chosen strength.*

In typical situations where SR comes into play one observes that – contrary to usual intuition – noise of a moderately chosen amplitude causes an increase of signal quality, thus supporting structural properties of a signal instead of smearing it and destroying information. This effect is measured in terms of the signal strength of an output signal with respect to the amplitude of the noisy perturbation. A more thorough explanation is given in Section 4.1.

Mathematically, the investigation of SR consists of a description of periodicity properties of diffusion trajectories. In a simplest situation, one may think of the one-dimensional diffusion

$$dX_t^\varepsilon = \left\{ -U'(X_t^\varepsilon) + A\sin\left(\frac{2\pi t}{T}\right)\right\}dt + \sqrt{\varepsilon}dW_t, \tag{4.1}$$

where U is a symmetric two-well potential, e.g. $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$, and W is a Brownian motion. The system (4.1) contains two types of perturbations of the deterministic system $\dot{x} = -U'(x)$. Firstly, the periodic *input signal* of amplitude $A > 0$ and period $T > 0$ leads to the effective time-dependent potential $U(t,x) = U(x) - Ax\sin\left(\frac{2\pi t}{T}\right)$.

Here, the amplitude A is assumed to be small, so that $U(t, \cdot)$ never degenerates into a one-well potential. Secondly, the *white noise perturbation* of small intensity $\varepsilon > 0$ enables transitions between the potential wells of U that would be impossible without noise.

For systems such as (4.1) one observes that, for certain parameter regimes of the noise amplitude ε and the input frequency T^{-1}, the trajectories of the diffusion X^ε exhibit 'quasi-periodic' behavior. This effect may roughly be explained as follows. The system (4.1) may be regarded as a small random perturbation of the deterministic system

$$\dot{x}(t) = -U'(t, x(t)). \tag{4.2}$$

Due to the small amplitude A of the periodic perturbation, the time-dependent potential $U(t, \cdot)$ possess two local minima at all times. The solutions of (4.2) are attracted by these (time-dependent) minima, so that the deterministic trajectory never leaves the potential well in which it starts. The addition of the stochastic forcing $\sqrt{\varepsilon} W$ to (4.2) enables transitions between the potential wells that would be energetically impossible without noise. For certain noise amplitudes, these *noise induced transitions* occur 'almost periodically' according to the period T, and the diffusion trajectory X^ε spends most of its time in the location of the energetically most favorable position, i.e. in the deeper one of the two potential wells. More precisely, there exists a noise amplitude $\varepsilon = \varepsilon(T)$ of moderate strength for which this periodicity property is most pronounced in some sense. The system is in *stochastic resonance*. The problem of *optimal tuning* consists in finding this optimal noise amplitude $\varepsilon(T)$, the *resonance point*. Equivalently, one may fix the noise amplitude and look for the corresponding period length $T(\varepsilon)$ that yields the best periodic response. Typically, the optimization problem of finding the resonance point can be solved only asymptotically in the *small noise limit* ($\varepsilon \to 0$), resp. the *large period limit* ($T \to \infty$).

Towards a mathematically precise understanding of SR, one needs to find a suitable notion of quantifying periodicity properties of diffusion trajectories, to give the problem of optimal tuning a precise meaning. Pavlyukevich [38] was the first to address this problem. He investigated physically meaningful measures for the quality of the periodic response, and showed that they exhibit certain unexpected drawbacks, which suggest to follow a different route. In this part of the thesis, we propose a probabilistic measure of quality that generalizes an approach of Herrmann and Imkeller [23] and overcomes the drawbacks of physical notions of periodicity.

This chapter provides an introducion to the concept of stochastic resonance and its historical development, and is organized as follows. In Section 4.1 we explain the basic idea of stochastic resonance by means of the nowadays classical toy-model of Nicolis and Benzi et al. Then we recall the essentials of Pavlyukevich's study of physical quality measures that led to our notion of periodicty (Section 4.2). In Section 4.3 we introduce our probabilistic concept. The subsequent chapters are devoted to the mathematical study of our approach.

4.1 The phenomenon of stochastic resonance

Stochastic resonance effects have been studied by physicists for about 20 years and are continually being discovered in numerous areas of natural sciences. Their first investigation originates in a stochastic toy model from climatology which may serve to explain some of its main features.

To give a qualitative explanation for the almost periodic recurrence of cold and warm ages (glacial cycles) in paleoclimatic data, Nicolis [36] and Benzi et al. [6] proposed a simple stochastic climate model based on an energy balance equation for the averaged global temperature $x(t)$ at time t. The balance between averaged absorbed and emitted radiative energy fluxes leads to a deterministic differential equation for $x(t)$ of the form

$$\begin{aligned} c\dot{x}(t) &= E_{\text{in}}(t) && - \; E_{\text{out}}(t) \\ &= Q(t)\Big\{1 - a(x(t))\Big\} && - \; \gamma x(t)^4. \end{aligned} \tag{4.3}$$

Here $E_{\text{in}} = Q\{1-a(x)\}$ and $E_{\text{out}} = \gamma x^4$ represent the absorbed and emitted radiative energy flux, respectively, and the constant c describes heat capacity. The outgoing energy γx^4 is modeled according to the Stefan-Boltzmann law for black body radiators, and γ is the Stefan constant. The incoming energy is described by two functions, the solar constant Q and the albedo a. The *solar constant* Q specifies the amount of solar energy that reaches the Earth's surface at time t. It fluctuates periodically at a very low frequency of 10^{-5} times per year due to periodic changes of the Earth's orbit's eccentricity (Milankovich cycles). Therefore, Q is supposed to be of the form

$$Q(t) = Q_0\Big(1 + A\sin\Big(\frac{2\pi t}{T}\Big)\Big). \tag{4.4}$$

Here the amplitude of the periodic variation is small, i.e. $A \approx 0.0001$. The *albedo function* $a = a(x)$ describes the yearly and globally averaged proportion of incoming energy that is reflected by the Earth's surface at surface temperature x, i.e. $1 - a(x)$ quantifies the proportion of absorbed solar energy, the relevant portion for the energy balance. In the simplest model (see Budyko [13] and Sellers [43]), one assumes that a takes essentially two values that correspond to the main surface temperature states: for low temperatures the reflection is high (due to a bright, frozen surface during cold ages), while for high temperatures the reflection is low (darker surface thanks to vegetation during warm ages). Between these two regimes, a is interpolated linearly.

Given these properties of the functions and parameters that enter the energy balance (4.3), one ends up with a deterministic differential equation whose essential properties may be subsumed in the following general setting. The r.h.s. of (4.3) may be written as the derivative of a time-dependent potential $U(t, x)$, so that (4.3) turns into an equation of the form

$$\dot{x}(t) = -U'\Big(\frac{t}{T}, x(t)\Big), \tag{4.5}$$

where $U'(t,x) = \frac{\partial}{\partial x}U(t,x)$ indicates the spatial derivative of U. Due to the small amplitude of the periodic perturbation (4.4), the potential function $U(t,\cdot)$ is a two-well potential at all times, i.e. it has three local extrema that correspond to equilibrium temperatures in the energy balance. There are two stable equilibrium solutions $x_\pm(t)$ of (4.5), which represent cold and warm climate states. They are approximately given by the local minima of the potential, since the derivative w.r.t. the time variable is negligible due to re-scaling with large T. The unstable one corresponding to the local maximum has no physical meaning and cannot be observed. Moreover, the periodic forcing term $Q(t)$ results in a periodically changing asymmetry of the potential. During each half-period, the left and the right potential well change the role of being the deeper one of the two. Physically, the latter is the energetically most favorable state.

So far, the simplified energy balance model (4.3) (resp. (4.5)) cannot picture reality in an adequate way. Depending on their initial positions, the solution trajectories $x(t)$ of (4.5) are attracted by one of the equilibrium positions $x_-(t)$ or $x_+(t)$, but each solution $x(t)$ gets stuck in one of the potential's wells for all times. Transitions between the two potential wells are energetically impossible, and consequently the model cannot account for a description of glacial cycles.

For that reason, Nicolis [36] and Benzi et al. [6] extended the energy balance by a stochastic forcing term. They proposed a stochastic energy balance, governed by the SDE

$$dX_t^\varepsilon = -U'\Big(\frac{t}{T}, X_t^\varepsilon\Big)dt + \sqrt{\varepsilon}dW_t, \quad X_0^\varepsilon = x_0, \tag{4.6}$$

where W is a Brownian motion, and $\sqrt{\varepsilon}$ quantifies the amplitude of the stochastic perturbation. In this equation negative values become possible, so one should think of U as being the primitive of the r.h.s. of (4.3) only for positive temperatures.

For bistable systems such as (4.6) the following observations are of crucial importance. Due to the addition of the noise term, transitions between the metastable states $x_\pm(t)$ of (4.5) become possible. Depending on the choice of parameters, the solution trajectories of (4.6) exhibit qualitatively different behavior.

If the noise amplitude $\sqrt{\varepsilon}$ is very small, the diffusion's dynamics is essentially governed by the deterministic dynamics, i.e. it spends

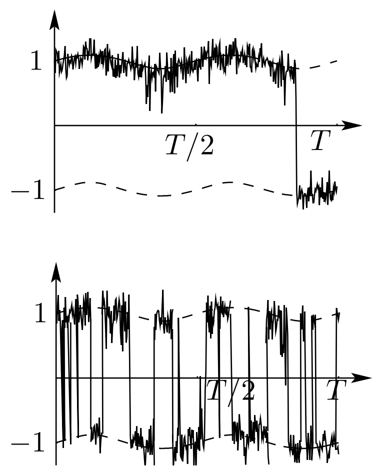

Figure 4.1: Small (t.) & large (b.) noise

most of the time in the vicinity of one of the potential wells, and only very rarely a transition to the other well occurs. For large noise amplitudes, the situation is completely different. In that case the stochastic perturbation dominates the picture, and the deterministic underlying geometry becomes almost invisble. This is illustrated in Figure 4.1 for the prototypical example of the potential $U(t,x) = U_0(x) - Ax \sin\left(\frac{2\pi t}{T}\right)$ corresponding to (4.1), where $U_0(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$.

There is, however, the following effect. At an *intermediate noise level*, the solution trajectories turn out to be 'almost periodic', i.e. the diffusion jumps rapidly between the wells each time they change the order of their depths, while in the time between these jumps it fluctuates around the respective local minimum of the well, see Figure 4.2.

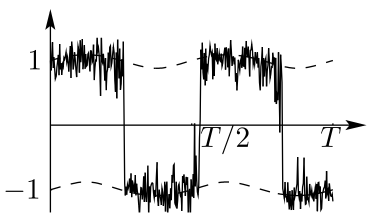

Figure 4.2: Optimal tuning

For the climate model (4.3) this means that, for a carefully chosen noise intensity, an additional stochastic perturbation may lead to a qualitative explanation of periodic transitions between cold and warm ages.

Let us restate this observation from a more abstract point of view. The above system may be regarded as follows. Besides the intrinsic geometry given by the potential U, it is driven by a *weak periodic input signal*, whose amplitude is too small to account for any visible effect in the output signal X^ε. The addition of noise makes the periodic input visible in the output signal, and for properly chosen noise of moderate intensity, this effect is well pronounced, i.e. the input signal is *amplified* by moderate noise. This effect is called *noise induced amplification*. The system is in *stochastic resonance* if this amplification effect is optimal in some sense to be made precise.

The stochastic resonance effect does not depend on the absolute value of the noise amplitude $\sqrt{\varepsilon}$, but on the proper relation of $\sqrt{\varepsilon}$ and the period length T of the periodic signal. The latter was fixed in the model, i.e. given by nature. In general, the noise intensity and the period T are linked via the relation $T = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ for some 'critical' value $\mu > 0$, which is a consequence of Kramers' law. This will be explained in more detail in the next section.

The crucial question that arises naturally in this situation is the following: how should the noise induced amplification effect be quantified, and how can one rigorously determine the *optimal tuning*, i.e. the noise intensity that provides the most pronounced amplification effect? For that purpose, one needs to define *quality measures* to measure periodicity of diffusion trajectories. The choice of an adequate quality measure is by no means a canonical task, as we shall see in the next section.

4.2 Physical notions of optimal tuning

The problem of optimal tuning was first addressed by Pavlyukevich. In his thesis [38], he investigates quality measures that are well known in the physical literature, and points out that the physicists' notions of optimal tuning suffer from certain defects. Specifically, he has shown that the most prominent physical notion of optimal tuning, the spectral power amplification, does not allow for a reduction of the diffusion dynamics to a simple model described by a two-state Markov chain that mimics only the hopping mechanism between the potential wells. In this section, we shall give a brief overview of his results, and compare the corresponding measures for the diffusion and the Markov chain. An excellent account of the topic is also given in [26] and [29].

4.2.1 Diffusion with small noise

In [38] the diffusion equation (4.6) is considered with a particularly simple time dependence of the potential $U(t,x)$. It is supposed to be time-space antisymmetric and piecewise constant w.r.t. time. More precisely, for a given time-independent double-well potential $U_1(x)$, the potential function U is defined in the strip $(t,x) \in [0,1) \times \mathbb{R}$ by

$$U(t,x) = \begin{cases} U_1(x) & \text{for } t \in [0, \frac{1}{2}), \\ U_1(-x) & \text{for } t \in [\frac{1}{2}, 1), \end{cases}$$

and periodically extended to the whole time axis via the relation $U(t+1,\cdot) = U(t,\cdot)$.

The double-well potential U_1 is supposed to have its local minima at ± 1 and a local maximum at 0. The extrema are supposed to be non-degenerate, i.e. with strictly positive (resp. negative) curvature.

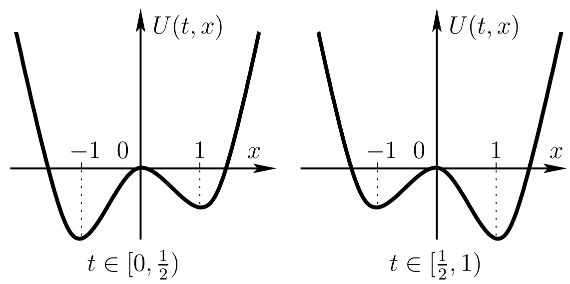

Figure 4.3: The potential $U(t,x)$

The diffusion X^ε possesses two intrinsic time scales that are related to the depths of the potential wells. Assume that $U_1(0) = 0$, $U_1(-1) = -\frac{V}{2}$ and $U_1(1) = -\frac{v}{2}$, where $v < V$. Then, according to Kramers' law, the exit times from domains of attraction for the time homogeneous diffusion

$$d\tilde{X}^\varepsilon_t = -U_1'(\tilde{X}^\varepsilon_t)dt + \sqrt{\varepsilon}dW_t, \quad \tilde{X}^\varepsilon_0 = x_0, \tag{4.7}$$

associated with U_1 are given to exponential order by $e^{V/\varepsilon}$ and $e^{v/\varepsilon}$, respectively. Since the diffusion X^ε is governed by an autonomous drift on each half-period, it is no surprise that these two time scales are also of central importance for the process X^ε. An exit from the shallow potential well of depth $\frac{v}{2}$ is possible only in exponential time scales larger than $e^{v/\varepsilon}$, hence one may expect to observe periodicity only in these scales. This was confirmed by Freidlin [18], who showed that the energy level v indeed serves as a treshold for periodicity properties. More precisely, he proved the following. Let $T = T^\varepsilon$ denote a time scale such that $\lim_{\varepsilon\to 0} \varepsilon \log T^\varepsilon = \mu$ exists. Then for $\mu < v$ the diffusion does not have enough time to leave either of the wells and stays in its initial well forever: for all $A > 0$ and $\delta > 0$

$$\boldsymbol{\lambda}\Big\{t \in [0, A] : |X^\varepsilon_{tT^\varepsilon} - \operatorname{sign}(x_0)| > \delta\Big\} \xrightarrow[\varepsilon\to 0]{} 0 \qquad \text{in } \mathbb{P}_{x_0}\text{-probability},$$

where $\boldsymbol{\lambda}$ denotes Lebesgue measure. On the other hand, if $\mu > v$, then for any $A > 0$ and $\delta > 0$

$$\boldsymbol{\lambda}\Big\{t \in [0, A] : |X^\varepsilon_{tT^\varepsilon} - \chi(t)| > \delta\Big\} \xrightarrow[\varepsilon\to 0]{} 0 \qquad \text{in } \mathbb{P}_{x_0}\text{-probability},$$

where χ describes the location of the energetically most favorable well:

$$\chi(t) = \begin{cases} -1 & \text{for } t \in [0, \frac{1}{2}), \\ +1 & \text{for } t \in [\frac{1}{2}, 1), \end{cases}$$

i.e. $\chi(t)$ is the coordinate of the global minimum of $U(t, \cdot)$. Thus, in case $\mu > v$, the diffusion exhibits *quasi-deterministic* behavior: for small ε, the diffusion is very close to the deterministic function χ.

Freidlin's result characterizes periodicity and explains why it may be observed only in exponential time scales that correspond to energy levels $\mu > v$, but his approach does not account for the 'quality' of periodicity, i.e. for an optimal tuning. It is intuitively clear that, as μ gets very large, periodicity is destroyed more and more. For $\mu > V$ one observes chaotic behavior, since the time scale is so large that transitions in both directions are possible during each half-period. So one clearly expects the optimal tuning to be comprised between v and V.

4.2.2 Reduction to a two-state Markov chain

According to the theory of stochastic resonance by McNamara and Wiesenfeld [33], the resonance behavior of the diffusion X^ε may be approximated via a finite state Markov chain that lives on the set of meta-stable states of X^ε. The underlying idea is roughly that periodic behavior of X^ε should essentially be determined by the transition mechanism between domains of attraction. This seems sound, since the diffusion – due to the small noise amplitude – mainly consists of small fluctuations around the underlying deterministic system, which in turn tends to one of the potential minima.

Only through a rare segment in the driving noise (a large deviation), an excursion to the other potential well may occur, and afterwards the trajectory again follows essentially the deterministic geometry. In this subsection, we shall describe the Markov chain that captures the diffusion's transitions, and present some of the results of [38] concerning its optimal tuning.

The reduced model may be set up as follows. We define a continuous time two-state Markov chain $\{Y_t^\varepsilon, t \geq 0\}$ that reflects the inter-well dynamics of X^ε on the set $\mathcal{S}^Y = \{-1, +1\}$ of meta-stable states of X^ε via its infinitesimal generator

$$G(t) = \begin{cases} G_1, & \text{if } t \ (\text{mod} 1) \in [0, \frac{1}{2}), \\ G_2, & \text{if } t \ (\text{mod} 1) \in [\frac{1}{2}, 1). \end{cases} \tag{4.8}$$

Here G_1 and G_2 are time-independent matrices of infinitesimal transition rates given by

$$G_1 = \begin{pmatrix} -\phi & \phi \\ \tilde{\phi} & -\tilde{\phi} \end{pmatrix} \qquad \text{and} \qquad G_2 = \begin{pmatrix} -\tilde{\phi} & \tilde{\phi} \\ \phi & -\phi \end{pmatrix}. \tag{4.9}$$

For Y to mimic the transition behavior of X, the rates ϕ and $\tilde{\phi}$ are chosen according to Kramers' law. Recall that the depths of the deep and the shallow well of U are given by $\frac{V}{2}$ and $\frac{v}{2}$, respectively. By Kramers' law the mean exit time of X from the potential wells is of the order $e^{V/\varepsilon}$, resp. $e^{v/\varepsilon}$. This suggests to define the transition rates for any $\varepsilon > 0$ by

$$\phi = pe^{-V/\varepsilon} \qquad \text{and} \qquad \tilde{\phi} = qe^{-v/\varepsilon} \tag{4.10}$$

with some sub-exponential pre-factors $p, q > 0$ the exact value of which is dispensable. The Markov chain Y is time-homogeneous on each half period, i.e. on each interval $[kT, (k+1)T)$, $k \geq 0$. The probabilities $p_{i,j}$ of jumping from i at time t to j between time t and $t + h$ satisfy

$$\begin{aligned} p_{-1,1}(t, h) &= \phi \cdot h + o(h), \\ p_{1,-1}(t, h) &= \tilde{\phi} \cdot h + o(h) \end{aligned}$$

in each 'even' half period, and similarly in the 'odd' half periods with the roles of ϕ and $\tilde{\phi}$ exchanged. In his thesis [38], Pavlyukevich determines the invariant measure of Y, i.e. the invariant measure of the homogenization of Y via state-space extension. This yields a time-dependent invariant (two-point) measure $\nu = \nu(t, \cdot)$ on $[0, \frac{1}{2}) \times \mathcal{S}^Y$ with boundary conditions to account for periodicity. Under this measure, he calculates (among others) the two physically most prominent quality measures for periodic tuning. The first one is the spectral power amplification coefficient

$$\eta^Y(\varepsilon, T) = \left| \int_0^1 \mathbb{E}_\nu \left[Y_{Ts}^{\varepsilon, T} \right] e^{2\pi i s} ds \right|^2, \tag{SPA}$$

which quantifies the power carried by the averaged trajectory of the Markov chain. The second one is the *signal-to-noise ratio*

$$\mathrm{SNR}(\varepsilon, T) = \frac{\eta^Y(\varepsilon, T)}{\varepsilon^2},$$

which measures this same averaged power, but this time relative to the intensity of the input signal. In the simple situation of the two-state Markov chain, the quantity $\eta^Y(\varepsilon, T)$ may be calculated explicilty, and one obtains

$$\eta^Y(\varepsilon, T) = \frac{4}{(V-v)^2} \frac{T^2(\phi - \psi)^2}{(\phi + \psi)^2 T^2 + \pi^2},$$

see [38], Proposition 4.2.1. Maximizing this expression and the corresponding one for $\mathrm{SNR}(\varepsilon, T)$ yields the following asymptotics. As $T \to \infty$, the SPA exhibits a local maximum at

$$\varepsilon(T) \approx \frac{v+V}{2} \frac{1}{\ln T}, \tag{4.11}$$

i.e. for large time scales T the optimal noise level is approximately given by (4.11). If one fixes the noise intensity and looks at time scales instead, one obtains in the small-noise limit $\varepsilon \to 0$

$$T(\varepsilon) \approx \frac{\pi}{\sqrt{2pq}} \sqrt{\frac{v}{V-v}} \exp\left\{\frac{V+v}{2\varepsilon}\right\}, \tag{4.12}$$

i.e. for small noise levels the most pronounced periodicity is obtained for time scales chosen according to (4.12). Similarly, one derives the following optimal noise intensities resp. optimal time scales for the signal-to-noise ratio. They are given by

$$\varepsilon(T) \approx \frac{v}{\ln T} \qquad \text{and} \qquad T(\varepsilon) \approx \frac{\pi\sqrt{v}}{q\sqrt{\varepsilon}} \exp\left\{\frac{v}{\varepsilon}\right\}. \tag{4.13}$$

See [38] for a precise formulation of the asymptotics (4.11), (4.12) and (4.13).

Let us summarize these observations. The above stated asymptotics show that both for the SPA and for the SNR an optimal tuning of the Markov chain exists, and the optimal time scales are given to exponential order by $\exp\left\{\frac{V+v}{2\varepsilon}\right\}$ and $\exp\left\{\frac{v}{\varepsilon}\right\}$, respectively. In particular, we see that the optimal tuning depends on the choice of the quality measure, and one may ask which one is the natural choice.

4.2.3 Spectral power amplification of the diffusion

Let us get back to the diffusion equation (4.6) with the piecewise constant potential described in Section 4.2.1. Following [38], we shall now describe the asymptotic behavior of the diffusion's SPA, and compare against the results of the previous subsection.

The diffusion X^ε is a time inhomogeneous Markov process, i.e. it has no invariant measure in the common sense. By its time-dependent invariant density $\nu = (\nu(t,\cdot), t \geq 0)$ we mean the invariant density (in the usual sense) of the space-extended time homogeneous Markov process $(X^\varepsilon_{tT},\, t \bmod 1)$. The random variable X^ε_{Tt} is then distributed according to $\nu(t,\cdot)$ under ν. The spectral power amplification (SPA) with respect to this equilibrium density is defined by

$$\eta^X(\varepsilon, T) = \left| \int_0^1 \mathbb{E}_\nu \left[X^\varepsilon_{Ts} \right] e^{2\pi i s} ds \right|^2 .$$

It describes the energy that the averaged trajectories carry on the frequency of the input signal. To optimize periodicity of X^ε w.r.t. the SPA means to seek a local maximum of $\varepsilon \mapsto \eta^X(\varepsilon, T)$. In [38] it was shown that such a local maximum surprisingly does not exist in the parameter regime where the Markov chain's SPA exhibits such one. Depending on whether $U_1^{(3)}(-1)$ is negative or positive, the diffusion's SPA is strictly decreasing resp. increasing w.r.t. ε in relevant time scales $T^\varepsilon \sim \exp\left\{\frac{\mu}{\varepsilon}\right\}$ with $\mu > v$. The SPA is therefore not suited for optimizing periodicity of the diffusion, and the idea of model reduction to the two-state Markov chain fails for the SPA. In view of the fact that the Markov chain described in the previous subsection is seen as a simple approximation of the diffusion, one is led to the erroneous imagination that the diffusion is also optimally tuned for the very same set of parameters ε and T.

The conclusion from these observations is that quality measures that rely on spectral properties of (averaged) paths are not appropriate for measuring periodicity properties of diffusions. The latter are, due to their stochastic nature, too irregular to exhibit periodicity in a strict sense. When considering quantities such as the SPA, all the small *intra-well fluctuations* the diffusion performs in the vicinity of the potential wells are taken into account. They are responsible for the unexpected behavior of the diffusion's SPA, and they are neglected when passing to the Markov chain.

For these reasons, one needs to look for measures of periodicity that are better suited to the fluctuating behavior of stochastic processes. This aim shall be addressed in the next section.

4.3 A probabilistic notion of optimal tuning

In view of Pavlyukevich's results, we propose a different quality measure that is purely probabilistic in its nature and neglects the small fluctuations that lead to the lack of robustness of physical measures. The approach we suggest here generalizes the results of Herrmann and Imkeller [23], who introduced exactly the same measure in the one-dimensional situation. It overcomes both of the drawbacks the physical measures suffer from: it yields an optimal tuning for the diffusion, and it is robust for the passage to the corresponding two-state Markov chain, i.e. both the Markov chain and the diffusion exhibit the same resonance picture.

In the general setting of finite dimensional diffusion processes, we study a dynamical system in d-dimensional Euclidean space perturbed by a d-dimensional Brownian motion W, i.e. we consider the solution of the stochastic differential equation

$$dX_t^\varepsilon = b\Big(\frac{t}{T^\varepsilon}, X_t^\varepsilon\Big)\, dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0. \tag{4.14}$$

One of the system's important features is that its inhomogeneity is weak in the sense that the drift depends on time only through a re-scaling by the large time scale parameter T^ε. It will be assumed to be exponentially large in ε, i.e. $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ for some $\mu > 0$, which corresponds to the situation in [23] and is motivated by the well known Kramers' law (Theorem 3.1). b is assumed to be one-periodic w.r.t. time. The deterministic system $\dot{\xi}_t = b(s, \xi_t)$ with *frozen* time parameter s is supposed to have two domains of attraction that do not depend on $s \geq 0$. In the 'classical' case of a drift derived from a potential, $b(t,x) = -\nabla_x U(t,x)$ for some potential function U, equation (4.14) describes the motion of a Brownian particle in a d-dimensional time inhomogeneous double well potential.

Since our stochastic resonance criterion shall capture only the transition mechanism between the two meta-stable sets of the system, our analysis relies on a suitable notion of transition or exit time. The Kramers-Eyring formula suggests to consider the energy unit μ corresponding to $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ as a natural measure of scale. Therefore, if at time s the system needs energy $e(s)$ to leave some meta-stable set, an exit from that set should occur at time

$$a_\mu = \inf\Big\{t \geq 0 : \ e(t) \leq \mu\Big\}$$

in the diffusion's natural time scale. If a_μ^i are the transition times for the two domains of attraction numbered $i = \pm 1$, we look at the probabilities of transitions between them within a time window $[(a_\mu^i - h)T^\varepsilon, (a_\mu^i + h)T^\varepsilon]$ for small $h > 0$. Assume for this purpose that the two corresponding meta-stable points are given by $x_i, i = \pm 1$, and denote by τ_ϱ^{-i} the random time at which the diffusion reaches the ϱ-neighborhood $B_\varrho(x_{-i})$ of x_{-i}. Then we use the following quantity to measure the quality of periodic tuning:

$$\mathcal{M}(\varepsilon, \mu) = \min_{i=\pm 1} \sup_{x \in B_\varrho(x_i)} \mathbb{P}_x\left(\tau_\varrho^{-i} \in [(a_\mu^i - h)T, (a_\mu^i + h)T]\right),$$

the minimum being taken in order to account for transitions back and forth. In order to exclude trivial or chaotic transition behavior, the scale parameter μ has to be restricted to an interval I_R of reasonable values which we call *resonance interval*.

With this measure of quality, the stochastic resonance point may be determined as follows. We first fix ε and the window width parameter $h > 0$, and maximize $\mathcal{M}(\varepsilon, \mu)$ in μ, eventually reached for the time scale $\mu_0(h)$. Then we call the eventually existing limit $\lim_{h\to 0} \mu_0(h)$ resonance point. To calculate $\mu_0(h)$ for fixed positive h we use large deviations techniques. In fact, our main result (Theorem 7.1) contains a formula which states that

$$\lim_{\varepsilon\to 0} \varepsilon \log \Big\{1 - \mathcal{M}(\varepsilon, \mu)\Big\} = \max_{i=\pm 1} \Big\{\mu - e_i(a_\mu^i - h)\Big\}.$$

This asymptotic relation holds uniformly w.r.t. μ on compact subsets of I_R, a fact which enables us to perform a maximization and find $\mu_0(h)$.

The techniques needed to prove our main result feature extensions and refinements of the fundamental large deviations theory for time homogeneous diffusions by Freidlin and Wentzell [19]. We prove a large deviations principle for the inhomogeneous diffusion (4.14) and strengthen this result to get uniformity in system parameters. Similarly to the time homogeneous case, where large deviations theory is applied to the problem of *diffusion exit* culminating in a mathematically rigorous proof of the Kramers-Eyring law, we study the problem of diffusion exit from a domain which is carefully chosen in order to allow for a detailed analysis of transition times. The main idea behind our analysis is that the natural time scale is so large that re-scaling in these units essentially leads to an asymptotic freezing of the time inhomogeneity, which has to be carefully studied, to hook up to the theory of large deviations of time homogeneous diffusions.

The forthcoming chapters are organized as follows. In Chapter 5 a large deviations principle for the weakly time inhomogeneous diffusion (4.14) is established (Proposition 5.4). The important feature of this LDP is its uniformity with respect to system parameters. This uniformity is crucial for the study of asymptotic exponential exit rates from domains of attraction, which are addressed in Chapter 6. Chapter 7 is concerned with developing the resonance criterion and computing the resonance point from the preceding results.

Chapter 5

Uniform large deviations for weakly inhomogeneous diffusions

For our understanding of stochastic resonance effects of diffusions with slow time dependence, we have to extend the large deviations results of Chapters 1 and 2 to diffusions moving in potential type landscapes with different valleys slowly and periodically changing their depths and positions. In this chapter, we provide this extension of the Freidlin-Wentzell theory, and restrict ourselves to the case of additive Brownian noise. This allows for the pathwise comparisons we shall employ, and which would not be available in the general case. In Section 5.1 we present a general result on large deviations for diffusions with weak time dependence, which exhibits the idea of the approach. In Section 5.2 we specialize this result to diffusions with a weak time dependence that originates in a slow periodic perturbation, and strengthen it to get uniformity in system parameters. These uniform large deviations will be our main tool for estimating the asymptotics of exit times in the next chapter.

5.1 A general result on weakly time inhomogeneous diffusions

In this subsection we shall extend the large deviations results of Freidlin and Wentzell to time inhomogeneous diffusions which are almost homogeneous in the small noise limit, so that in fact we are able to compare to the large deviation principle for time homogeneous diffusions. The result presented here is not strong enough for the treatment of stochastic resonance as one needs uniformity in the system's parameters, but it most clearly exhibits the idea underlying the technical arguments of the next section.

Consider the family X^ε, $\varepsilon > 0$, of solutions of the SDE

$$dX_t^\varepsilon = b^\varepsilon(t, X_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, dW_t, \qquad t \geq 0,\ X_0^\varepsilon = x_0 \in \mathbb{R}^d\,. \tag{5.1}$$

We assume that (5.1) admits a global strong solution for all $\varepsilon > 0$. Our main large deviations result for diffusions for which time inhomogeneity fades out in the small noise limit is summarized in the following Proposition.

5.1 Proposition (Large deviations principle)**.** *Assume that the drift of the SDE* (5.1) *satisfies*

$$\lim_{\varepsilon \to 0} b^\varepsilon(t, x) = b(x) \tag{5.2}$$

for all $t \geq 0$*, uniformly w.r.t.* x *on compact subsets of* $\mathbb{R}^d$*, for some locally Lipschitz function* $b : \mathbb{R}^d \to \mathbb{R}^d$*. Moreover, assume that the time homogeneous diffusion* Y^ε *governed by the SDE*

$$dY_t^\varepsilon = b(Y_t^\varepsilon)\, dt + \sqrt{\varepsilon}\, dW_t, \qquad t \geq 0,\ Y_0^\varepsilon = x_0 \in \mathbb{R}^d\,.$$

associated to the limiting drift b *does not explode.*

Then (X^ε) *satisfies a large deviations principle on any finite time interval* $[0, T]$ *with good rate function* $I_{0T}^{x_0}$ *given by* (1.12)*.*

Proof. For notational convenience, we drop the ε-dependence of X and Y. We shall prove that X and Y are exponentially equivalent, i.e. for any $\delta > 0$ we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(\rho_{0T}(X, Y) > \delta\right) = -\infty. \tag{5.3}$$

In order to verify this, fix some $\delta > 0$, and observe that

$$\|X_t - Y_t\| \leq \int_0^t \|b^\varepsilon(u, X_u) - b(X_u)\|\, du + \int_0^t \|b(X_u) - b(Y_u)\|\, du.$$

For $R > 0$ let $\tau_R := \inf\{t \geq 0 : X_t \notin B_R(x_0)\}$, let $\tilde{\tau}_R$ be defined similarly with X replaced by Y, and $\sigma_R := \tau_R \wedge \tilde{\tau}_R$. The local Lipschitz continuity of b implies the existence of some constant $K_R(x_0)$ such that $\|b(x) - b(y)\| \leq K_R(x_0)\|x - y\|$ for $x, y \in B_R(x_0)$. An application of Gronwall's Lemma A.2 yields

$$\rho_{0T}(X, Y) \leq e^{K_R(x_0)T} \int_0^T \|b^\varepsilon(u, X_u) - b(X_u)\|\, du \quad \text{ on } \{\sigma_R > T\}.$$

Due to uniform convergence, for any $\eta > 0$ we can find some $\varepsilon_0 > 0$ s.t.

$$\sup_{x \in B_R(x_0)} \|b^\varepsilon(t, x) - b(x)\| \leq \eta \quad \text{ for } t \in [0, T],\ \varepsilon < \varepsilon_0.$$

This implies

$$\rho_{0T}(X, Y) \leq \eta T e^{K_R(x_0)T} \quad \text{ for } \varepsilon < \varepsilon_0 \text{ on } \{\sigma_R > T\}. \tag{5.4}$$

By choosing η small enough s.t. $\rho_{0T}(X,Y) \le \delta/2$ on $\{\sigma_R > T\}$ (i.e. X and Y are very close before they exit from $B_R(x_0)$), we see that for $\varepsilon < \varepsilon_0$

$$\mathbb{P}\left(\rho_{0T}(X,Y) > \delta\right) \le \mathbb{P}(\tau_R \le T) + \mathbb{P}(\tilde{\tau}_R \le T).$$

Since X and Y are close within the ball $B_R(x_0)$, we deduce that if X escapes from $B_R(x_0)$ before time T, then Y must at least escape from $B_{R/2}(x_0)$ before time T (if $R > \delta$). So we have

$$\mathbb{P}\left(\rho_{0T}(X,Y) > \delta\right) \le \mathbb{P}(\tilde{\tau}_{R/2} \le T)$$

for $\varepsilon < \varepsilon_0$. Hence the LDP for Y gives

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\rho_{0T}(X,Y) > \delta\right) \le -\inf\left\{V(x_0,y,t) :\ 0 \le t \le T, \|y - x_0\| \ge R/2\right\}.$$

Sending $R \to \infty$ yields the desired result (see Theorem 4.2.13 in [17]). $\square$

It is easy to see that the uniformity of the LDP w.r.t. the diffusion's initial condition also holds for the weakly inhomogeneous process X^ε of this proposition. One only has to carry over Proposition 5.6.14 in [17], which is easily done using some Gronwall argument. Then the proof of the uniformity is the same as in the time homogeneous case (see [17], Corollary 5.6.15). We omit the details.

5.2 Weak inhomogeneity through slow periodic variation

In this section we shall deal with a particular class of diffusions for which the drift term is subject to a very slow periodic time inhomogeneity. More precisely, we shall be concerned with solutions of the following SDE taking values in d-dimensional Euclidean space, driven by a d-dimensional Brownian motion W of intensity ε:

$$dX_t^\varepsilon = b\Big(\frac{t}{T^\varepsilon}, X_t^\varepsilon\Big)dt + \sqrt{\varepsilon}\, dW_t, \quad t \ge 0,\ X_0^\varepsilon = x_0 \in \mathbb{R}^d\,. \tag{5.5}$$

In accordance with Kramers' law, T^ε is taken to be an exponentially large time scale, i.e.

$$T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\} \quad \text{with } \mu > 0. \tag{5.6}$$

The drift $b(t,x)$ of (5.5) is supposed to be a time-periodic function of period one. Concerning its regularity properties, we suppose it to be locally Lipschitz in both variables, i.e. for $R > 0$, $x \in \mathbb{R}^d$ there are constants $K_R(x)$ and $\kappa_R(x)$ such that

$$\|b(t,y_1) - b(t,y_2)\| \ \le\ K_R(x)\, \|y_1 - y_2\|\,, \tag{5.7}$$

$$\|b(t,y) - b(s,y)\| \ \le\ \kappa_R(x)\, |t - s| \tag{5.8}$$

for all $y, y_1, y_2 \in B_R(x)$ and $s, t \geq 0$. Furthermore, we shall assume that the drift term forces the diffusion to stay in compact sets for long times in order to get sufficiently 'small' level sets. We suppose that there are constants η, $R_0 > 0$ such that

$$\langle x, b(t,x) \rangle < -\eta \, \|x\| \tag{5.9}$$

for $t \geq 0$ and $\|x\| \geq R_0$. This condition is stronger than (2.2), so the existence of a unique strong and non-exploding solution is guaranteed by Corollary 2.2. Moreover, by Corollary 2.5, this growth condition implies that the diffusion is concentrated in a compact set with high probability (see also Theorem 2.3 for the precise asymptotics).

5.2.1 Properties of the quasi-potential

Taking large period limits in the subsequently derived large deviations results for our diffusions with slow periodic variation will require to freeze the time parameter in the drift term. The corresponding rate functions are given a separate treatment in this subsection. We shall briefly discuss their regularity properties.

For $s \geq 0$, $T > 0$ we consider

$$I_{0T}^s(\varphi) = \begin{cases} \frac{1}{2} \int_0^T \|\dot{\varphi}_t - b(s, \varphi_t)\|^2 \, dt, & \text{if } \varphi \text{ is absolutely continuous,} \\ +\infty, & \text{otherwise.} \end{cases} \tag{5.10}$$

We shall also employ the associated *cost functions*. For $s \geq 0$, $x, y \in \mathbb{R}^d$ they are given by

$$V^s(x,y,t) = \inf \Big\{ I_{0t}^s(\varphi) : \, \varphi \in C_{0t}, \varphi_0 = x, \varphi_t = y \Big\}. \tag{5.11}$$

$V^s(x,y,t)$ is the cost of forcing the *frozen* system

$$dY_t^\varepsilon = b(s, Y_t^\varepsilon) \, dt + \sqrt{\varepsilon} \, dW_t, \quad t \geq 0,$$

to be at the point y at time t when starting at x. The corresponding *quasi-potential*

$$V^s(x,y) = \inf_{t>0} V^s(x,y,t) \tag{5.12}$$

describes the cost for the frozen system to go from x to y eventually. Let us note that since the drift b is locally Lipschitz in the time variable, the family of action functionals I_{0T}^s is continuous w.r.t. the parameter s, and the corresponding cost functions and pseudo-potentials inherit this continuity property. Let us recall some further useful properties of the quasi-potentials and their underlying cost and rate functions. The following properties are immediate.

5.2 Lemma. *For any $x, y, z \in \mathbb{R}^d$ and $s, t, u \geq 0$ we have* *a)* $V^s(x, y, t + u) \le V^s(x, z, t) + V^s(z, y, u)$,

b) $(s, y) \mapsto V^s(x, y, t)$ *is continuous on* $\mathbb{R}_+ \times \mathbb{R}^d$,

c) $\inf_{\|y\| \ge R} V^s(x, y, t) \xrightarrow[R \to \infty]{} \infty$ *uniformly w.r.t.* $s \ge 0$.

The following lemma establishes the local Lipschitz continuity of the quasi-potential w.r.t. the state variables, uniformly w.r.t. the parameter s.

5.3 Lemma. *For any compact set* $K \subset \mathbb{R}^d$ *there exists* $\Gamma_K \ge 0$ *such that*

$$\sup_{s \ge 0} V^s(x, y) \;\le\; \Gamma_K \, \|x - y\| \,, \qquad x, y \in K.$$

Proof. Let $x, y \in K$, and set $T = \|x - y\|$. There exists some radius $R > 0$ such that $K \subset B_R(0)$. We construct a path $\varphi \in C_{0T}$ from x to y by setting

$$\varphi_t = x + \frac{y - x}{\|y - x\|}\, t, \qquad t \in [0, T].$$

Since $b(s, \cdot)$ is locally Lipschitz, uniformly w.r.t. $s \ge 0$, we obtain an upper bound for the energy of φ as follows:

$$\begin{aligned}
I^s_{0T}(\varphi) \;&\le\; \frac{1}{2} \sup_{u \ge 0} \int_0^T \|\dot{\varphi}_t - b(u, \varphi_t)\|^2 dt \\
&\le\; \frac{1}{2} \int_0^T \Big(1 + \sup_{0 \le u \le 1} \|b(u, \varphi_t)\|\Big)^2 dt \\
&\le\; \frac{1}{2} \int_0^T \Big(1 + \kappa_R(0) + \|b(0, \varphi_t)\|\,\Big)^2 dt \\
&\le\; \frac{1}{2} \int_0^T \Big(1 + \kappa_R(0) + K_R(0)\, \|\varphi_t\| + \|b(0, 0)\|\,\Big)^2 dt \\
&\le \frac{T}{2} \Big(1 + \kappa_R(0) + R K_R(0) + \|b(0, 0)\|\,\Big)^2.
\end{aligned}$$

For $\Gamma_K := \frac{1}{2} \Big(1 + \kappa_R(0) + R K_R(0) + \|b(0, 0)\|\,\Big)^2$ and by definition of T, we obtain

$$\sup_{s \ge 0} V^s(x, y) \;\le\; \sup_{s \ge 0} I^s_{0T}(\varphi) \;\le\; \Gamma_K \, \|x - y\| \,. \qquad \square$$

5.2.2 Large deviations

We shall now specialize the general large deviations results of Section 5.1 to the family $(X^\varepsilon)_{\varepsilon > 0}$ of solutions of (5.5). At the same time they will be strengthened, to obtain uniformity w.r.t. to the system's parameters: the scale parameter μ, the starting time, and the initial condition.

It is an immediate consequence of Proposition 5.1 that the solution of (5.5) satisfies a large deviations principle with rate function I^0_{0T}, i.e. the rate function coincides with the one of the time homogeneous diffusion governed by the frozen drift $b(0,\cdot)$. In order to see this, one only has to mention that $\lim_{\varepsilon\to 0} b\Big(\frac{t}{T^\varepsilon},x\Big) = b(0,x)$ locally uniformly w.r.t. x due to the Lipschitz assumptions on b. To prove uniformity w.r.t. the above mentioned system parameters, we have to refine the comparison arguments involved in the proof of Proposition 5.1. Moreover, we shall consider exponentially large starting times, which – through the scaling with T^ε – result in arbitrary autonomous drift terms $b(s,\cdot)$ with $s \geq 0$.

The diffusion (5.5) is a time inhomogeneous Markov process. The solution starting at time $r \geq 0$ with initial condition $x \in \mathbb{R}^d$ has the same law as the solution $X^{r,x}$ of the SDE

$$dX^{r,x}_t = b\Big(\frac{r+t}{T^\varepsilon}, X^{r,x}_t\Big)dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0,\ X^{r,x}_0 = x \in \mathbb{R}^d . \tag{5.13}$$

We denote its law by $\mathbb{P}_{x,r}(\cdot)$, and fix $T \geq 0$. The following proposition states our main result about large deviations for the diffusion (5.5).

5.4 Proposition. *Let $K \subset \mathbb{R}^d$ be a compact set and $\mathcal{V} \subset (0,\infty)$. For $\mu \in \mathcal{V}$, $r \in [0,1]$ and $\beta \geq 0$ let $S^{r,\beta}(\varepsilon,\mu)$ be a neighborhood of rT^ε such that*

$$\limsup_{\varepsilon\to 0} \sup_{\mu\in\mathcal{V}, r\in[0,1]} \frac{\operatorname{diam}(S^{r,\beta}(\varepsilon,\mu))}{T^\varepsilon} \leq \beta.$$

Then for any closed $F \subset C_{0T}$, there exists $\delta = \delta(F)$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in K, \mu\in\mathcal{V}, u\in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in F) \ \leq -\inf_{y\in K} \inf_{\varphi\in F^{\gamma_0}, \varphi_0 = y} I^r_{0T}(\varphi)$$

where $\gamma_0 = \gamma_0(F) = \beta\delta(F)$ and F^{γ_0} is the closed γ_0-neighborhood of F.
For any open $G \subset C_{0T}$, there exists $\delta = \delta(G)$ and $\beta_0 = \beta_0(G)$ such that, if $\beta \leq \beta_0$,

$$\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{y\in K, \mu\in\mathcal{V}, u\in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in G) \ \geq -\sup_{y\in K} \inf_{\varphi\in G^{\gamma_0}, \varphi_0 = y} I^r_{0T}(\varphi),$$

where $\gamma_0 = \gamma_0(G) = \beta\delta(G)$ and G^{γ_0} is the complement of $(G^c)^{\gamma_0}$.
These bounds hold uniformly w.r.t. r.

5.5 Remark. The upper bound means that for any $\vartheta > 0$ we can find $\varepsilon_0 > 0$ s.t. for $\varepsilon \leq \varepsilon_0$ we have

$$\varepsilon \log \sup_{y\in K, \mu\in\mathcal{V}, u\in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in F) \leq -\inf_{y\in K} \inf_{\varphi\in F^{\gamma_0}, \varphi_0 = y} I^r_{0T}(\varphi) + \vartheta.$$

The uniformity in the statement means that ε_0 can be chosen independently of r. A similar statement holds for the lower bound.

Observe that the expression for the blowup-factor $\gamma_0(F)$ depends on the set F only through $\delta(F)$ which is independent of β, and that $\gamma_0(F) \to 0$ as $\beta \to 0$ for all F. In particular, if β is equal to zero, we recover the classical bound of the uniform LDP.

Proof of Proposition 5.4. For $y \in \mathbb{R}^d$ and $r \in [0,1]$ let $Y^{r,y}$ be the solution of the autonomous SDE

$$dY_t^{r,y} = b(r, Y_t^{r,y})\,dt + \sqrt{\varepsilon}\,dW_t, \quad t \geq 0, \quad Y_0^{r,y} = y.$$

Let $\mathcal{W} \subset [0,1]$ and $r_0 \in \mathcal{W}$. For $r \in \mathcal{W}$, $u \in S^{r,\beta}(\varepsilon,\mu), \mu \in \mathcal{V}$ and $R > 0$ let $\tau_R^{u,y} := \inf\{t \geq 0 : X_t^{u,y} \notin B_R(0)\}$, and let $\tilde{\tau}_R^{r_0,y}$ be defined similarly with $X^{u,y}$ replaced by $Y^{r_0,y}$, and $\sigma_R^{u,y,r_0} := \tau_R^{u,y} \wedge \tilde{\tau}_R^{r_0,y}$.
As a consequence of Gronwall's lemma we see just as in the proof of Proposition 5.1 that for $r, r_0 \in [0,1], u \in S^{r,\beta}(\varepsilon,\mu)$

$$\rho_{0T}(X^{u,y}, Y^{r_0,y}) \leq e^{K_R(0)T} \int_0^T \left\| b\Big(\frac{u+t}{T^\varepsilon}, X_t^{u,y}\Big) - b(r_0, X_t^{u,y}) \right\| dt \quad \text{ on } \{\sigma_R^{u,y,r_0} > T\}.$$

This implies

$$\rho_{0T}(X^{u,y}, Y^{r_0,y}) \leq \kappa_R(0) T e^{K_R(0)T} \left(\frac{\operatorname{diam}(S^{r,\beta}(\varepsilon,\mu)) + T}{T^\varepsilon} + |r - r_0| \right)$$

on $\{\sigma_R^{u,y,r_0} > T\}$. Due to our assumption the last expression is bounded by

$$\beta_1 = \beta_1(\mathcal{W}) = \beta_0(\mathcal{W}) M(R) \quad \text{ as } \varepsilon \to 0, \tag{5.14}$$

where $\beta_0(\mathcal{W}) := \beta + \sup_{r \in \mathcal{W}} |r - r_0|$ and $M(R) := T\kappa_R(0) e^{K_R(0)T}$.
Upper bound: Fix some closed set $F \subset C_{0T}$. For all $\gamma > 0$ we have

$$\mathbb{P}(X^{u,y} \in F) \leq \mathbb{P}(Y^{r_0,y} \in F^\gamma) + \mathbb{P}(\rho_{0T}(X^{u,y}, Y^{r_0,y}) > \gamma).$$

This yields

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in K, \mu \in \mathcal{V}, r \in \mathcal{W}, u \in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in F) \\
&\quad \leq \limsup_{\varepsilon \to 0} \varepsilon \log \max \Big\{ \sup_{y \in K} \mathbb{P}(Y^{r_0,y} \in F^\gamma), \\
&\qquad\qquad \sup_{y \in K, \mu \in \mathcal{V}, r \in \mathcal{W}, u \in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}(\rho_{0T}(X^{u,y}, Y^{r_0,y}) > \gamma) \Big\}.
\end{aligned}$$

Now we wish to find some γ such that the asymptotics of the maximum is determined by the left term $\sup_{y \in K} \mathbb{P}(Y^{r_0,y} \in F^\gamma)$. In that case the uniform LDP for Y (Corollary 2.10) will give us the bound

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in K, \mu \in \mathcal{V}, r \in \mathcal{W}, u \in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in F) \leq - \inf_{y \in K} \inf_{\varphi \in F^\gamma,\, \varphi_0 = y} I_{0T}^{r_0}(\varphi). \tag{5.15}$$

Unfortunately, such a γ will depend on F. In order to see that it exists and can be chosen as claimed in the statement, we define

$$\Theta(R,\varepsilon) := \sup_{r \in [0,1], y \in K, \mu \in \mathcal{V}, u \in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}(\tau_R^{u,y} \leq T) + \sup_{r \in [0,1], y \in K} \mathbb{P}(\tilde{\tau}_R^{r,y} \leq T).$$

By Theorem 2.3 and Remark 2.4 we have

$$\limsup_{\varepsilon\to 0} \varepsilon \log \Theta(R,\varepsilon) \le -\eta R$$

for all $R \ge R_1$. Hence we may fix $R \ge R_1$ such that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \Theta(R,\varepsilon) \le -\sup_{r\in[0,1]} \inf_{y\in K} \inf_{\varphi\in F,\varphi_0=y} I^r_{0T}(\varphi).$$

Let $\delta(F) = M(R)$, and note that $\delta(F)$ is independent of β and $\mathcal{W}$. By (5.14), for any $\gamma > \beta_1(\mathcal{W}) = \beta_0(\mathcal{W})\delta(F)$ we can find $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$

$$\sup_{r\in\mathcal{W},y\in K,\mu\in\mathcal{V},u\in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}(\rho_{0T}(X^{u,y}, Y^{r_0,y}) > \gamma) \le \Theta(R,\varepsilon). \tag{5.16}$$

Hence, for $\gamma > \beta_1(\mathcal{W})$, the definition of $\delta(F)$ implies

$$\begin{aligned} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{r\in\mathcal{W},y\in K,\mu\in\mathcal{V},u\in S^{r,\beta}(\varepsilon,\mu)} \mathbb{P}(\rho_{0T}(X^{u,y}, Y^{r_0,y}) > \gamma) \ &\le \ \limsup_{\varepsilon\to 0} \varepsilon \log \Theta(R,\varepsilon) \\ \le -\sup_{r\in[0,1]} \inf_{y\in K} \inf_{\varphi\in F,\varphi_0=y} I^r_{0T}(\varphi) \ &\le \ -\inf_{y\in K} \inf_{\varphi\in F^\gamma,\varphi_0=y} I^{r_0}_{0T}(\varphi), \end{aligned}$$

which implies (5.15). The particular choice $\mathcal{W} = \{r_0\}$ yields this bound for all $\gamma > \gamma_0(F) = \beta\delta(F)$ given in the statement and proves the claimed bound. By taking the limit $\gamma \to \gamma_0(F)$ we get the asserted upper bound since $I^{r_0}_{0T}$ is a good rate function (see [17], Lemma 4.1.6).

It remains to prove the uniformity w.r.t. r. For that purpose fix $\vartheta > 0$, and for $r_0 \in [0,1]$ let $\mathcal{W}_{r_0}$ be a neighborhood of r_0. By the continuity of $r \mapsto I^r_{0T}$ and Lemma 4.1.6 in [17] we can assume $\mathcal{W}_{r_0}$ to be small enough such that for $r \in \mathcal{W}_{r_0}$, denoting $\gamma^* = \beta_0(\mathcal{W}_{r_0})\delta(F)$ and $\gamma_0 = \beta\delta(F)$,

$$\inf_{y\in K} \inf_{\varphi\in F^{\gamma^*},\,\varphi_0=y} I^{r_0}_{0T}(\varphi) \ \ge \ \inf_{y\in K} \inf_{\varphi\in F^{\gamma_0},\,\varphi_0=y} I^{r_0}_{0T}(\varphi) - \vartheta/4 \ \ge \ \inf_{y\in K} \inf_{\varphi\in F^{\gamma_0},\,\varphi_0=y} I^{r}_{0T}(\varphi) - \vartheta/2.$$

Due to compactness we can choose finitely many points $r_1, ..., r_N$ such that their corresponding neighborhoods cover $[0,1]$. Denote $\gamma^*_n := \beta_0(\mathcal{W}_{r_n})\delta(F)$. For each $1 \le n \le N$ there exists some $\varepsilon_n > 0$ such that for $\varepsilon \le \varepsilon_n$ and $r \in \mathcal{W}_{r_n}$,

$$\begin{aligned} \varepsilon \log \sup_{y\in K,\mu\in\mathcal{V},u\in S^r(\varepsilon,\mu)} \mathbb{P}_{y,u}(X^\varepsilon \in F) \ &\le \ -\inf_{y\in K} \inf_{\varphi\in F^{\gamma^*_n},\varphi_0=y} I^{r_n}_{0T}(\varphi) + \frac{\vartheta}{2} \\ &\le \ -\inf_{y\in K} \inf_{\varphi\in F^{\gamma_0},\varphi_0=y} I^{r}_{0T}(\varphi) + \vartheta. \end{aligned}$$

Hence for $\varepsilon \le \min_{1\le n\le N} \varepsilon_n$, the preceding inequality holds for all $r \in [0,1]$.

Lower bound: Let $G \subset C_{0T}$ be an open set. Consider the increasing function

$$f(l) := \frac{1}{\eta} \sup_{y\in K} \inf_{\varphi\in G^l:\ \varphi_0=y} I^{r_0}_{0T}(\varphi),$$

let $l_0 = \inf\{l \geq 0 : \ f(l) = +\infty\}$, and recall that η is the constant introduced in the growth condition for the drift.
Assume first that $l_0 < \infty$ (this is guaranteed if G is bounded), and set

$$R := f\Big((l_0 - \beta_0(\mathcal{W})) \vee \frac{l_0}{2}\Big) \vee R_1 \quad \text{and} \quad \gamma := \beta_0(\mathcal{W})M(R),$$

where R_1 is given by Theorem 2.3. Then

$$\mathbb{P}(Y^{r_0,y} \in G^\gamma) \leq \mathbb{P}(X^{u,y} \in G) + \mathbb{P}(\rho_{0T}(Y^{r_0,y}, X^{u,y}) > \gamma).$$

By the uniform LDP for $Y^{r_0,y}$ and (5.16) we conclude that

$$\begin{aligned}
-\eta f(\gamma) = & -\sup_{y\in K} \inf_{\varphi\in G^\gamma,\, \varphi_0 = y} I^{r_0}_{0T}(\varphi) \\
\leq & \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{y\in K} \mathbb{P}(Y^{r_0,y} \in G^\gamma) \\
\leq & \max\Big\{ \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W}, y\in K, \mu\in\mathcal{V}, u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y} \in G), \\
& \qquad \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{r\in\mathcal{W}, y\in K, \mu\in\mathcal{V}, u\in S^r(\varepsilon,\mu)} \mathbb{P}(\rho_{0T}(Y^{r_0,y}, X^{u,y}) > \gamma)\Big\} \\
\leq & \max\Big\{ \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W}, y\in K, \mu\in\mathcal{V}, u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y} \in G), \\
& \qquad \limsup_{\varepsilon\to 0} \varepsilon \log \Theta(R,\varepsilon)\Big\}.
\end{aligned}$$

Since f is increasing and $R \geq R_1$, we obtain by Theorem 2.3

$$\begin{aligned}
-\eta f(\gamma + \beta_0(\mathcal{W})) \ & \leq \ -\eta f(\gamma) \\
& \leq \ \max\Big\{ \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W}, y\in K, \mu\in\mathcal{V}, u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y} \in G), \ -\eta R\Big\}.
\end{aligned}$$

Now we have to compare $f(\gamma)$ and R in order to see when this maximum is given by the left term.
If $f(\gamma) > R$, then $\gamma > (l_0 - \beta_0(\mathcal{W})) \vee \frac{l_0}{2} \geq l_0 - \beta_0(\mathcal{W})$ by monotonicity of f, hence $f(\gamma + \beta_0(\mathcal{W})) = +\infty$ by definition of l_0. Otherwise we have $f(\gamma) \leq R$, which means that the left term dominates the maximum.
In both cases we get

$$-\eta f(\gamma + \beta_0(\mathcal{W})) \ \leq \ \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W}, y\in K, \mu\in\mathcal{V}, u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y} \in G).$$

Now consider the unbounded case $l_0 = +\infty$. Let $\beta_0(G) := \sup_{l\geq 0} \frac{l}{M(f(l))}$, the existence of which was claimed in the statement. If $\beta_0(\mathcal{W}) < \beta_0(G)$, we can choose l_1 such that

$\frac{l_1}{M(f(l_1))} \geq \beta_0(\mathcal{W})$ and set $\gamma := \beta_0(\mathcal{W})M(f(l_1))$. Using the same arguments as in the bounded case, we deduce that

$$-\eta f(\gamma) \;\leq\; \max\Big\{\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W},y\in K,\mu\in\mathcal{V},u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y}\in G), -\eta f(l_1)\Big\}.$$

Since f is increasing and $l_1 \geq \gamma$ we obtain

$$-\eta f(\gamma) \;\leq\; \liminf_{\varepsilon\to 0} \varepsilon \log \inf_{r\in\mathcal{W},y\in K,\mu\in\mathcal{V},u\in S^r(\varepsilon,\mu)} \mathbb{P}(X^{u,y}\in G).$$

In both the bounded and the unbounded case we have found $\gamma = \beta_0(\mathcal{W})\delta(G)$ such that the desired bound holds: we have $\delta(G) = M(R)+1$ in the bounded case and $\delta(G) = M(f(l_1))$ in the unbounded case. Furthermore, the choice $\mathcal{W} = \{r_0\}$ corresponds to $\beta_0(\mathcal{W}) = \beta$ and yields $\gamma_0(G) = \beta\delta(G)$, in complete analogy to the situation of the upper bound. The uniformity is also proved in exactly the same way as already shown for the upper bound. $\square$

Chapter 6

Exit and entrance times of domains of attraction

The aim of this chapter is to determine the exact small noise asymptotics of exit times from certain carefully chosen domains for the weakly and periodically perturbed system (5.5). This is achieved by using suitable splittings of exponentially long time intervals, which allow us to hook up to the large deviations results of the previous chapter. The obtained asymptotics allow for maximizing transition rates between domains of attraction of the diffusion and constitute the main ingredient for our probabilistic notion of stochastic resonance treated in the next chapter.
In Section 6.1 we introduce some properties concerning the underlying deterministic geometry of (5.5), which essentially state that this system resembles the geometry of a two-well potential, although it does not need to be conservative. The two remaining sections are devoted to the proofs of upper and lower bounds for the asymptotic exponential rate in the main result of this chapter (Theorem 6.3) on the transition time asymptotics.

6.1 Geometric preliminaries

We continue to study asymptotic properties of diffusions with weakly periodic drifts given by the SDE

$$dX_t^\varepsilon = b\Big(\frac{t}{T^\varepsilon}, X_t^\varepsilon\Big)\, dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0,\ X_0^\varepsilon = x_0 \in \mathbb{R}^d\,. \tag{6.1}$$

We assume as before that b satisfies the local Lipschitz conditions (5.7) and (5.8), and that the growth of the inward drift is sufficiently strong near infinity which is expressed by (5.9). The latter ensures that the diffusion will asymptotically be concentrated on a compact set, and this holds on an exponential scale (see Corollary 2.5).

To study the effects of weak periodicity of the drift on the asymptotic behavior of exit times from domains of attraction, we shall make the following simple assumptions on the geometry of b, namely its attraction and conservation properties. Essentially, we shall assume that $\mathbb{R}^d$ is split into two domains of attraction, separated by a simple geometric boundary which is invariant in time. The additional conditions concerning the geometry of b are as follows.

6.1 Assumption. *The d-dimensional ordinary differential equation*

$$\dot{\varphi}_s(t) = b(s, \varphi_s(t)), \quad t \geq 0, \tag{6.2}$$

admits two stable equilibria x_- and x_+ in $\mathbb{R}^d$ which do not depend on $s \geq 0$. Moreover, the domains of attraction defined by

$$A_\pm(s) = \left\{ y \in \mathbb{R}^d : \ \varphi_s(0) = y \quad \textit{implies} \quad \lim_{t\to\infty} \varphi_s(t) = x_\pm \right\} \tag{6.3}$$

are also independent of $s \geq 0$ and denoted by $A_\pm$. They are supposed to satisfy $\overline{A_- \cup A_+} = \mathbb{R}^d$, and $\partial A_- = \partial A_+$. We denote by χ the common boundary.

This assumption could be weakened. We could let the stable equilibrium points and the domains of attraction depend on $s \geq 0$. The asymptotic results concerning the exit and entrance time remain true in this more general setting. We stick to Assumption 6.1 for reasons of notational simplicity.

According to the Kramers-Eyring law (Theorem 3.1), the mean time a homogeneous diffusion of noise intensity ε needs to leave a potential well of depth $\frac{v}{2}$ is of the order $\exp\left\{\frac{v}{\varepsilon}\right\}$. Nature therefore imposes the time scales $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$, $\mu > 0$, which we have to employ to observe transitions between the domains of attraction. For simplicity we measure these scales in their corresponding energy units μ.

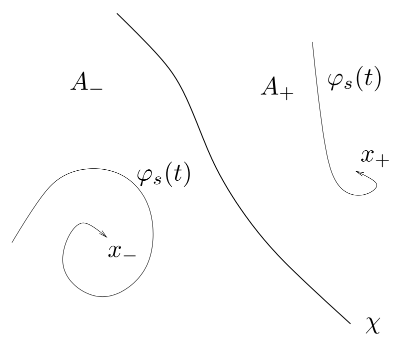

Figure 6.1: Domains of attraction

The main subject of investigation in this section is given by the exit times of the domains of attraction $A_\pm$, provided that the weakly time inhomogeneous diffusion starts near the equilibrium points $x_\pm$. By obvious symmetry reasons, we may restrict our attention to the case of an exit from A_-.

As we shall show, this exit time depends on the quasi-potential, that is on the cost function taken on the set of all functions starting in the neighborhood of x_- and exiting the domain of attraction through χ. For this reason we introduce the one-periodic *energy function*

$$e(s) := \inf_{y\in\chi} V^s(x_-, y) < \infty \quad \text{ for } s \geq 0, \tag{6.4}$$

which is continuous on $\mathbb{R}_+$. In the gradient case $b(t,x) = -\nabla_x U(t,x)$, this function coincides with twice the depth of the potential barrier to be overcome in order to exit from A_-, i.e. the energy the diffusion needs to leave A_-. Therefore scales μ – corresponding to the Kramers-Eyring times $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ according to the chosen parametrization – at which we expect transitions between the domains of attraction must be comprised between

$$\mu_* := \inf_{t\geq 0} e(t) \qquad \text{and} \qquad \mu^* := \sup_{t\geq 0} e(t).$$

These two constants are finite and are reached at least once per period since $e(t)$ is continuous and periodic. Now fix a time scale parameter μ. This parameter serves as a threshold for the energy, and we expect to observe an exit from A_- at the first time t at which $e(t)$ falls below μ. For $\mu \in]\mu_*, \mu^*[$ we therefore define

$$a_\mu = \inf\{t \geq 0 : \ e(t) \leq \mu\}, \qquad \alpha_\mu = \inf\{t \geq 0 : \ e(t) < \mu\}. \tag{6.5}$$

The subtle difference between a_μ and α_μ may be important, but we shall rule it out for our considerations by making the following assumption.

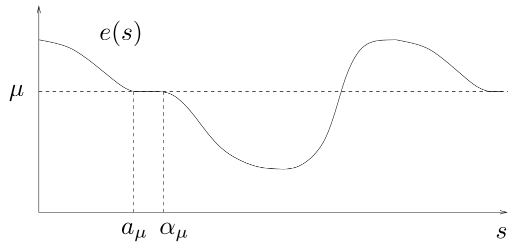

Figure 6.2: Definition of a_μ and α_μ

6.2 Assumption. *The energy function $e(t)$ is strictly monotonous between its (discrete) extremes, and every local extremum is global.*

Under this assumption we have $a_\mu = \alpha_\mu$. We are now in a position to state the main result of this section. Let $\varrho > 0$ be small enough such that the Euclidean ball $B_\varrho(x_+) \subset A_+$, and let us define the first entrance time into this ball by

$$\tau_\varrho = \inf\{t \geq 0 : \ X_t^\varepsilon \in B_\varrho(x_+)\}. \tag{6.6}$$

This stopping time depends of course on ε, but for notational simplicity we suppress this dependence.

6.3 Theorem. *Let $\mu < e(0)$. There exist $\eta > 0$ and $h_0 > 0$ such that for $h \leq h_0$*

$$\lim_{\varepsilon\to 0} \varepsilon \log \sup_{y\in B_\eta(x_-)} \mathbb{P}_y\left(\tau_\varrho \notin [(a_\mu - h)T^\varepsilon, (a_\mu + h)T^\varepsilon]\right) \ = \ \mu - e(a_\mu - h).$$

Moreover, this convergence is uniform w.r.t. μ on compact subsets of $]\mu_*, e(0)[$.

Note that Assumption 6.2 implies the continuity of $\mu \mapsto \mu - e(a_\mu - h)$. The statement of the theorem may be paraphrased in the following way. It specifies time windows in which transitions between the domains of attraction will be observed with very high probability. In particular, if $e(t)$ is strictly monotonous between its extremes, we prove that the entrance time into a neighborhood of x_+ will be located near $a_\mu T^\varepsilon$ in the small noise limit. The assumption $\mu < e(0)$ is only a technical assumption in order to avoid instantaneous jumping of the diffusion to the other valley. It can always be achieved by simply starting the diffusion a little later. We could even assume that $e(0) = \mu^*$ which then would yield uniform convergence on compact subsets of $]\mu_*, \mu^*[$.

The rest of this chapter is devoted to the proof of this main result and is subdivided into separate sections in which lower and upper bounds are treated.

6.2 Lower bound for the exit rate

We have to establish upper and lower bounds on the transition time τ_ϱ which both should be exceeded with an exponentially small probability that has to be determined exactly. It will turn out that the probability of exceeding the upper bound $(\alpha_\mu + h)T^\varepsilon$ vanishes asymptotically to all exponential orders, so the exact large deviations rate shall be determined only by the probability $\mathbb{P}_x\left(\tau_\varrho \leq (a_\mu - h)T^\varepsilon\right)$ of exceeding the lower bound.
Deriving a lower bound of the latter probability and an upper bound of the probability $\mathbb{P}_x\left(\tau_\varrho \geq (\alpha_\mu + h)T^\varepsilon\right)$ consists essentially of the same problem. In both cases one has to prove large deviations type upper bounds for probabilities of the form $\mathbb{P}_x\left(\tau_\varrho \geq s(\varepsilon)\right)$ for suitably chosen $s(\varepsilon)$. This can be expressed in terms of the problem of *diffusion exit* from a carefully chosen bounded domain.

Recall that τ_ϱ is the *first entrance time* of a small neighborhood $B_\varrho(x_+)$ of the equilibrium point x_+. Consider for $R, \varrho > 0$ the bounded domain

$$D = D(R, \varrho) := \overline{B_R(0)} \setminus B_\varrho(x_+),$$

and let

$$\tau_D := \inf\{t \geq 0 : \ X_t \notin D\}$$

be the *first exit time* of X from D. An exit from D means that either X enters $B_\varrho(x_+)$, i.e. we have a transition to the other equilibrium, or X leaves $B_R(0)$. But, as a consequence of our growth condition (5.9), the probability of the latter event does not contribute on the large deviations scale due to Theorem 2.3, as the following simple argument shows.

Let $s(\varepsilon, \mu) = sT^\varepsilon$ for some $s > 0$. Since $\tau_D = \tau_\varrho \wedge \sigma_R$ where σ_R is the time of the diffusion's first exit from $B_R(0)$, Theorem 2.3 provides constants $R_1, \varepsilon_1 > 0$ s.t. for

$R \geq R_1$, $\varepsilon \leq \varepsilon_1$, and $\|x\| \leq R/2$

$$\begin{aligned} \mathbb{P}_x\left(\tau_\varrho \geq s(\varepsilon,\mu)\right) &\leq \mathbb{P}_x\left(\{\tau_\varrho \geq s(\varepsilon,\mu)\} \cap \{\sigma_R \geq s(\varepsilon,\mu)\}\right) + \mathbb{P}_x\left(\sigma_R < s(\varepsilon,\mu)\right) \\ &\leq \mathbb{P}_x\left(\tau_D \geq s(\varepsilon,\mu)\right) + C\eta^2\, \frac{s(\varepsilon,\mu)}{\varepsilon}\, e^{-\frac{\eta R}{\varepsilon}}. \end{aligned} \tag{6.7}$$

By the choice of $s(\varepsilon,\mu)$ and $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$, the right term in the last sum is of the order $\frac{1}{\varepsilon}\exp\left\{\frac{\mu-\eta R}{\varepsilon}\right\}$, i.e. it can be assumed to be exponentially small of any exponential order required by choosing R suitably large. Obviously, this holds uniformly with respect to μ on compact sets. This argument shows that the investigation of asymptotic properties of the laws of τ_ϱ may be replaced by a study of similar properties of τ_D, with an error that may be chosen arbitrarily small by increasing R.

Similarly to the time homogeneous exit problem, we need a lemma which shows how to approximate the energy of a transition by the cost along particular trajectories which exit some neighborhood of D. This is of central importance to the estimation of the asymptotic law of τ_D.

6.4 Lemma. *Let $\vartheta > 0$ and M a compact interval of $\mathbb{R}_+$. Then there exist $T_0 > 0$ and $\delta > 0$ with the following property:*
For all $x \in D$ and $s \in M$, we can find a continuous path $\zeta^{x,s} \in C_{0,T_0}$ starting in $\zeta_0^{x,s} = x$ and ending at some point of distance $d(\zeta_{T_0}^{x,s}, D) \geq \delta$ away from D such that

$$I_{0T_0}^s(\zeta^{x,s}) \leq e(s) + \vartheta \quad \text{for all} \quad s \in M.$$

Proof. This proof extends arguments presented in Lemma 5.7.18 and 5.7.19 in [17]. Fix $\vartheta > 0$, and let us decompose the domain D into three different ones. Fixing $l > 0$, define a domain β_l by

$$\beta_l = \{x \in D : \ \mathrm{dist}(x,\chi) < l\}.$$

We recall that χ is the separation between A_- and A_+. Then we define two closed sets $D_- = (D \setminus \beta_l) \cap A_-$ and $D_+ = (D \setminus \beta_l) \cap A_+$. We shall construct appropriate paths from points $y \in D$ to points a positive distance away from D not exceeding the energy $e(s)$ by more than ϑ uniformly in $s \in M$ in four steps.

Step 1. Assume first that $y \in D_-$. For $l > 0$ small enough we construct $\delta_1^l > 0, S_1^l > 0$ and a path $\psi_1^{s,y,l}$ defined on a time interval $[0, \tau_1^{s,y,l}]$ with $\tau_1^{s,y,l} \leq S_1^l$ for all $y \in D_-, s \in M$ and along which we exit a δ_1^l–neighborhood of D_- at cost at most $e(s) + \frac{2}{3}\vartheta$.

Step 1.1. In a first step we go from y to a small neighborhood $B_l(x_-)$ of x_-, in time at most $T_1^l < \infty$, without cost.

We denote by $\varphi_1^{s,y,l}$ the trajectory starting at $\varphi_1^{s,y,l}(0) = y \in D_-$ of

$$\dot\varphi_1(t) = b(s, \varphi_1(t)),$$

and reaching $B_l(x_-)$ at time $\sigma_1^{y,s,l}$. Since $D_- \subset A_-$ and due to Assumption 6.1, $\sigma_1^{y,s,l}$ is finite. Moreover, since b is locally Lipschitz, stability of solutions with respect to initial conditions and smooth changes of vector fields implies that there exist open neighborhoods W_y of y and $\mathcal{W}_s$ of s and $T_1^{s,y,l} > 0$ such that, for all $z \in W_y$, $u \in \mathcal{W}_s$, $\sigma_1^{u,z,l} \leq T_1^{s,y,l}$. Recall that D_- is compact. Therefore we may find a finite cover of $D_- \times M$ by such sets, and consequently $T_1^l < \infty$ such that for all $y \in D_-$ and $s \in M$, $\sigma_1^{s,y,l} \leq T_1^l$. Denote $z^{s,y,l} = \varphi_1^{s,y,l}(\sigma_1^{s,y,l})$.

Step 1.2. In a second step, we go from a small neighborhood $B_l(x_-)$ of x_- to the equilibrium point x_-, in time at most 1, at cost at most $\frac{\vartheta}{3}$.
In fact, by the continuity of the cost function, for l small enough, $s \in M$, there exists a continuous path $\varphi_2^{s,y,l}$ of time length $\sigma_2^{s,y,l} \leq 1$ such that $\varphi_2^{s,y,l}(0) = z^{s,y,l}$, $\varphi_2^{s,y,l}(\sigma_2^{s,y,l}) = x_-$ and $I_{0\sigma_2^{s,y,l}}(\varphi_2^{s,y,l}) \leq \vartheta/3$.

Step 1.3. In a third step, we exit some δ–neighborhood of D_-, starting from the equilibrium point x_-, in time at most $T_3 < \infty$, at cost at most $e(s) + \frac{\vartheta}{3}$ for $s \in M$.
By (6.4) and the continuity of the cost function for any $s \in M$ there exists $z_s \notin A_- \supset D_-$, $T_3^s < \infty$, some neighborhood $\mathcal{W}_s$ of s and for $u \in \mathcal{W}_s$ we have $\varphi_3^u \in C_{0\sigma_3^u}$ such that $\varphi_3^u(0) = x_-$, $\varphi_3^u(\sigma_3^u) = z_s$, $\sigma_3^u \leq T_3^s$ and

$$\sup_{u \in \mathcal{W}_s} I^u_{0\sigma_3^u}(\varphi_3^u) \leq e(s) + \vartheta/3.$$

Use the compactness of M to find a finite cover of M by such neighborhoods, and thus some $T_3 < \infty$ such that all the statements hold with $\sigma_3^s \leq T_3$ for all $s \in M$. Finally remark that the exit point is at least a distance $\delta = \inf_{i \in J} |z_i|$ away from the boundary of D_-, if $z_i, i \in J$, are the exit points corresponding to the finite cover.

In order to complete Step 1, we now define a path $\psi_1^{s,y,l} \in C_{0\tau_1^{s,y,l}}$ by concatenating $\varphi_1^{s,y,l}$, $\varphi_2^{s,y,l}$ and φ_3^s. This way, for small $l > 0$ we find $S_1^l > 0$ such that for all $s \in M, y \in D_-$ we have $\tau_1^{s,y,l} \leq S_1^l$, $\psi_1^{s,y,l}(\tau_1^{s,y,l}) = y$, $\psi_1^{s,y,l}(\tau_1^{s,y,l}) \notin A_-$ and

$$I^s_{0\tau_1^{s,y,l}}(\psi_1^{s,y,l}) \;\leq\; e(s) + \tfrac{2}{3}\vartheta \quad \text{for all} \quad s \in M, y \in D_-.$$

At this point we can encounter two cases. In the first case $\psi_1^{s,y,l}$ exits a δ_l-neighborhood of $B_R(0)$. In this case we continue with Step 4. In the second case, $\psi_1^{s,y,l}$ exits D_- into β_l, and we continue with Step 2.

Step 2. For l small enough, we start in $y \in \beta_l$, to construct $S_2^l > 0$ and a path $\psi_2^{s,y,l}$ defined on a time interval $[0, \tau_2^{s,y,l}]$ with $\tau_2^{s,y,l} \leq S_2^l$ for all $y \in D_-, s \in M$ and along which we exit β_l into the interior of D_+ at cost at most $\frac{\vartheta}{3}$.
In fact, due to the continuity of the cost function (see Lemma 5.2), there exists $l > 0$ small enough such that for any $s \in M, y \in \overline{\beta_l}$ there exists $z_{s,y,l}$ in the interior of D_+, such that $\psi_2^{s,y,l}(0) = y$, $\psi_2^{s,y,l}(\tau_2^{s,y,l}) = z_{s,y,l}$ and $I^u_{0\tau_2^{s,y,l}}(\psi_2^{s,y,l}) \leq \vartheta/3$. We may take

$S_2^l = 1$.

Step 3. We start in $y \in D_+$, to construct $\delta_3^l > 0, S_3^l > 0$ and a path $\psi_3^{s,y,l}$ defined on a time interval $[0, \tau_3^{s,y,l}]$ with $\tau_3^{s,y,l} \leq S_3^l$ for all $y \in D_-, s \in M$ and along which we exit D_+ into $B_{\varrho - \delta_3^l}(x_+)$ at no cost.

Let $\delta_3^l = \varrho/2$. Since D_+ is compact and contained in the domain of attraction of x_+, stability of the solutions of the differential equation $\dot{\varphi}(t) = b(s, \varphi(t))$ with respect to the initial condition $y \in D_+$ and the parameter s guarantees the existence of some time $S_3^l > 0$ such that the entrance time $\tau_3^{s,y,l}$ of $B_{\varrho/2}(x_+)$ by the solution starting in y is bounded by S_3^l. Therefore we may take $\psi_3^{s,y,l}$ to be defined by this solution restricted to the time interval before its entrance into $B_{\varrho/2}(x_+)$.

Step 4. For $l > 0$ small enough we start in $x \in D_-$ and construct $T_0 > 0, \delta > 0$ and a path $\zeta^{s,x}$ defined on the time interval $[0, T_0]$, exiting a δ-neighborhood of D at cost at most $e(s) + \vartheta$ for all $s \in M$.

For l small enough, take $T_0 = S_1^l + S_2^l + S_3^l$. We just have to concatenate paths constructed in the first three steps. Recall that $\psi_1^{s,x,l}$ passes through the equilibrium x_- due to Step 1. In case $\psi_1^{s,x,l}$ exits a δ_1^l-neighborhood of $B_R(0)$, just let the path spend enough time in x_- without cost to obtain a path $\zeta^{s,x,l}$ defined on $[0, T_0]$, and take $\delta = \delta_1^l$. In the other case, we concatenate three paths constructed in Steps 1 - 3, to obtain a path defined on a subinterval of $[0, T_0]$ depending on s, x, l and which exits a δ_3^l–neighborhood of D. Recall from step 1 that this path also passes through x_-. It remains to redefine the path by spending extra time at no cost in this equilibrium point, to complete the proof. □

We now proceed to the estimation of uniform lower bounds for the asymptotic law of τ_D. The uniformity has to be understood in the sense of Remark 5.5.

6.5 Proposition. *Let K be a compact subset of D.*

a) If $e(s) > \mu$, then

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in K} \mathbb{P}_x(\tau_D < sT^\varepsilon) \ \geq \ \mu - e(s),$$

locally uniformly on $\{(s, \mu) : \ \mu_* < \mu < \min(e(0), e(s)), 0 \leq s \leq 1\}$.

b) If $e(s) < \mu$, then

$$\lim_{\varepsilon \to 0} \varepsilon \log \sup_{x \in K} \mathbb{P}_x(\tau_D \geq sT^\varepsilon) \ = \ -\infty,$$

locally uniformly on $\{(s, \mu) : \ e(s) < \mu < e(0), 0 \leq s \leq 1\}$.

Proof. We choose a compact subset L of $[0, 1]$ and a compact subset M of $]\mu_*, e(0)[$ as well as some $\vartheta > 0$ such that

$$|e(s) - \mu| \geq \vartheta \quad \forall (s, \mu) \in L \times M.$$

Later on we shall assume that $e(s)-\mu$ is uniformly positive resp. negative in order to prove a) resp. b).

In a first step, we apply Lemma 6.4 to approximate the energy function $e(s)$ by the rate function along a particular path, uniformly w.r.t. s. For the chosen ϑ it yields $T_0>0$ and $\delta>0$ as well as continuous paths $\zeta^{x,s}$ indexed by $x\in D$ and $s\in[0,1]$ ending a distance at least δ away from D such that for all $x\in D$ and $s\in[0,1]$

$$I^s_{0T_0}(\zeta^{x,s})\le e(s)+\frac{\vartheta}{4}.$$

In a second step, we use the Markov property to estimate the probability of exiting D after time sT^ε by a large product of exit probabilities after time intervals of length independent of ε and μ. Since for $\varepsilon>0$, $\mu\in M$ the interval $[0,sT^\varepsilon]$ becomes arbitrarily large as $\varepsilon\to0$, we introduce a splitting into intervals of length $\nu\ge T_0$ independent of ε and μ. For $k\in\mathbb{N}_0$ let $t_k=t_k(s,\varepsilon,\mu):=sT^\varepsilon-k\nu$. Then we have for $k\in\mathbb{N}_0$ and $x\in D$

$$\begin{aligned}\mathbb{P}_x\left(\tau_D\ge t_k\right) &= \mathbb{E}_x\left(\mathbf{1}_{\{\tau_D\ge t_k\}}\mathbf{1}_{\{\tau_D\ge t_{k+1}\}}\right) = \mathbb{E}_x\left(\mathbf{1}_{\{\tau_D\ge t_{k+1}\}}\,\mathbb{E}\left[\mathbf{1}_{\{\tau_D\ge t_k\}}\middle|\mathscr{F}_{t_{k+1}}\right]\right)\\ &\le \mathbb{P}_x\left(\tau_D\ge t_{k+1}\right)\sup_{y\in D}\mathbb{P}_{y,t_{k+1}}\left(\tau_D\ge\nu\right)\end{aligned}$$

Here $\mathbb{P}_{y,s}$ denotes the law of $X^{s,y}$, defined by the SDE

$$dX^{s,y}_t=b\Big(\frac{s+t}{T^\varepsilon},X^{s,y}_t\Big)\,dt+\sqrt{\varepsilon}\,dW_t,\quad t\ge0,\ X^{s,y}_0=y\in\mathbb{R}^d\,.$$

On intervals $[0,\nu]$ it coincides with the law of the original process X^ε on $[s,s+\nu]$, but of course paths may differ. Denoting $q_k(s,\varepsilon,\mu):=\sup_{y\in D}\mathbb{P}_{y,t_k}(\tau_D\ge\nu)$, an iteration of the latter argument yields

$$\sup_{x\in K}\mathbb{P}_x\left(\tau_D\ge sT^\varepsilon\right)\le\prod_{k=1}^{N(\varepsilon,\mu)}q_k(s,\varepsilon,\mu)\tag{6.8}$$

whenever $N(\varepsilon,\mu)\,\nu<sT^\varepsilon$.

For the further estimation of the q_k we apply some LDP to the product (6.8). This relies on the following idea. If we choose $N(\varepsilon,\mu)$ of the order $\varepsilon T^\varepsilon$, then the starting times t_k appearing in the product belong to some neighborhood of sT^ε that, compared to T^ε, shrinks to a point asymptotically. Consequently, the family of diffusions underlying the product (6.8) is uniformly exponentially equivalent to the homogeneous diffusion governed by the drift $b(s,\cdot)$. This will be done in the following third step.

For $x\in D$, $s\in[0,1]$ let

$$\Psi(x,s):=\left\{\psi\in C_{0T_0}:\ \rho_{0,T_0}(\psi,\zeta^{x,s})<\tfrac{\delta}{2}\right\}$$

be the open $\delta/2$-neighborhood of the path chosen in the first step, and let

$$\Psi(x) := \bigcup_{s\in[0,1]} \Psi(x,s).$$

To apply our large deviations estimates in this situation, note first that conditions concerning τ_D translate into constraints for the trajectories of X^ε as figuring in the preceding section: due to the definition of $\Psi(x,s)$, the choice $\nu \geq T_0$ and Lemma 6.4 we know that for $y \in D, k \leq N(\varepsilon,\mu)$, if $X^{t_k,y}$ belongs to $\Psi(x)$, then for sure $X^{t_k,y}$ exits D before time ν. Keeping this in mind, we may apply Proposition 5.4 to the neighborhoods

$$S^{s,0}(\varepsilon,\mu) = \Big[sT^\varepsilon - \nu N(\varepsilon,\mu), sT^\varepsilon + \nu\Big]$$

of sT^ε. Each of the intervals $[t_k, t_k+\nu]$ is contained in $S^{s,0}(\varepsilon,\mu)$. As mentioned before, $N(\varepsilon,\mu)$ is chosen of the order $\varepsilon T^\varepsilon$, and this can be done uniformly w.r.t. $\mu \in M$. More precisely, we assume to have constants $0 < c_1 < c_2$ such that $c_1\varepsilon T^\varepsilon \leq N(\varepsilon,\mu) \leq c_2\varepsilon T^\varepsilon$. Then

$$\lim_{\varepsilon\to 0} \sup_{s\in[0,1],\, \mu\in M} \frac{\operatorname{diam} S^{s,0}(\varepsilon,\mu)}{T^\varepsilon} = 0,$$

and by the uniform large deviations principle of Proposition 5.4 we obtain the lower bound

$$\begin{aligned}\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{y\in K,\, \mu\in M,\, k\leq N(\varepsilon,\mu)} \mathbb{P}_{y,t_k}(\tau_D < \nu) \ &\geq\ -\sup_{y\in K} \inf_{\psi\in\Psi(y)} I^s_{0T_0}(\psi)\\ &\geq\ -\sup_{y\in K} I^s_{0T_0}(\zeta^{y,s})\ \geq\ -e(s) - \frac{\vartheta}{4}.\end{aligned}$$

We stress that this bound is uniform w.r.t. s in the sense of Remark 5.5, so we can find $\varepsilon_0 > 0$ independent of s such that for $\varepsilon \leq \varepsilon_0$, $\mu \in M$ and $k \leq N(\varepsilon,\mu)$

$$\begin{aligned}1 - q_k(s,\varepsilon,\mu) \ &=\ \inf_{y\in D} \mathbb{P}_{y,t_k}(\tau_D < \nu)\\ &\geq\ \inf_{y\in D,\, \mu\in M,\, j\leq N(\varepsilon,\mu)} \mathbb{P}_{y,t_j}(\tau_D < \nu)\ \geq\ \exp\Big\{-\frac{1}{\varepsilon}\Big(e(s) + \frac{\vartheta}{2}\Big)\Big\}.\end{aligned}$$

From this we obtain

$$\begin{aligned}\sup_{x\in K} \mathbb{P}_x\Big(\tau_D \geq sT^\varepsilon\Big) &\leq \prod_{k=1}^{N(\varepsilon,\mu)} q_k(s,\varepsilon,\mu) \leq \Big(1 - \exp\Big\{-\frac{1}{\varepsilon}\Big(e(s) + \frac{\vartheta}{2}\Big)\Big\}\Big)^{N(\varepsilon,\mu)}\\ &= \exp\Big\{N(\varepsilon,\mu)\log\Big(1 - \exp\Big\{-\frac{1}{\varepsilon}\Big(e(s) + \frac{\vartheta}{2}\Big)\Big\}\Big)\Big\} =: m(\varepsilon,\mu).\end{aligned}$$

Since $\log(1-x) \leq -x$ for $0 \leq x < 1$ we have

$$m(\varepsilon,\mu)\ \leq\ \exp\Big\{-c_1\varepsilon\ \exp\Big\{\frac{\mu}{\varepsilon} - \frac{1}{\varepsilon}\Big(e(s) + \frac{\vartheta}{2}\Big)\Big\}\Big\}. \tag{6.9}$$

In the fourth and last step, we exploit this bound of $m(\varepsilon, \mu)$ to obtain the claimed asymptotic bounds.

In order to prove a), assume that $\mu < e(s)$ for $(s, \mu) \in L \times M$. Then the inner exponential in (6.9) approaches 0 on $L \times M$. Using the inequality $1 - e^{-x} \geq x \exp(-1)$ on $[0, 1]$, we conclude that there exists $\varepsilon_1 \in (0, \varepsilon_0)$ such that for all $\varepsilon \leq \varepsilon_1$ and $(s, \mu) \in L \times M$

$$\begin{aligned}
\varepsilon \log \inf_{x \in K} \mathbb{P}_x \Big(\tau_D < sT^\varepsilon \Big) &\geq \varepsilon \log \Big(1 - m(\varepsilon, \mu) \Big) \\
&\geq \varepsilon \log \Big(\varepsilon c_1 \exp(-1) \ \exp \Big\{ \frac{1}{\varepsilon} \Big(\mu - e(s) - \frac{\vartheta}{2} \Big) \Big\} \Big) \\
&= -\varepsilon + \varepsilon \log c_1 + \varepsilon \log \varepsilon \ + \mu - e(s) - \frac{\vartheta}{2} \\
&\geq \mu - e(s) - \vartheta.
\end{aligned}$$

For b) assume $\mu > e(s)$ on $L \times M$. Then

$$\begin{aligned}
\varepsilon \log \sup_{x \in K} \mathbb{P}_x \Big(\tau_D \geq sT^\varepsilon \Big) &\leq \varepsilon \log m(\varepsilon, \mu) \\
&\leq -c_1 \varepsilon \exp \Big\{ -\frac{1}{\varepsilon} \Big(\mu - e(s) - \frac{\vartheta}{2} \Big) \Big\} \xrightarrow[\varepsilon \to 0]{} -\infty. \qquad \Box
\end{aligned}$$

As a consequence of these large deviations type results on the asymptotic distribution of τ_D and the remarks preceding the statement of Lemma 6.4 and Proposition 6.5, we get the following asymptotics for the transition time of the diffusion.

6.6 Proposition. *Let $x \in A_-$. There exists $h_0 > 0$ such that for $h \leq h_0$*

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(\tau_\varrho \leq (a_\mu - h)T^\varepsilon) \ \geq \ \mu - e(a_\mu - h), \tag{6.10}$$

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{P}_x(\tau_\varrho \geq (\alpha_\mu + h)T^\varepsilon) \ = \ -\infty. \tag{6.11}$$

Moreover, these convergence statements hold uniformly w.r.t. x on compact subsets of D and w.r.t. μ on compact subsets of $]\mu_*, e(0)[$.

Proof. As the estimation (6.7) based on Theorem 2.3 at the beginning of the section shows, we may derive the required estimates for τ_D instead of τ_ϱ, if R is chosen large enough.

Let M be a compact subset of $]\mu_*, e(0)[$. Then $0 < a_\mu < 1$ for $\mu \in M$, which yields the existence of $h_0 > 0$ such that the compact set $L_h := \{a_\mu - h : \ \mu \in M\}$ is contained in $]0, 1[$ for $h \leq h_0$. Moreover, we have $e(s) > \mu$ for $0 < s < a_\mu$ due to the assumptions on e, uniformly w.r.t. $(s, \mu) \in L_h \times M$ by the continuity of e. Hence by Proposition 6.5 a)

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x \in K} \mathbb{P}_x(\tau_D \leq sT^\varepsilon) \ \geq \ \mu - e(s),$$

uniformly on $L_h \times M$ for all $h \leq h_0$. By setting $s = a_\mu - h$ we obtain the first asymptotic inequality. The second one follows in a completely analoguous way from Proposition 6.5 b) since $e(\alpha_\mu + h) < \mu$ for small enough h. $\square$

6.3 Upper bound for the exit rate

Let us next derive upper bounds for the exponential exit rate which resemble the lower bounds just obtained. We need an extension of a result obtained by Freidlin and Wentzell (Lemma 5.4 in [48]).

6.7 Lemma. *Let K be a compact subset of $A_- \setminus \{x_-\}$. There exist $T_0 > 0$ and $c > 0$ such that for all $T \geq T_0$, $s \in [0,1]$ and for each $\varphi \in C_{0T}$ taking its values in K we have*

$$I^s_{0T}(\varphi) \;\geq\; c(T - T_0).$$

Proof. Let $\phi_{s,x}$ be the solution of the differential equation

$$\dot{\phi}_{s,x}(t) = b(s, \phi_{s,x}(t)), \quad \phi_{s,x}(0) = x \in K.$$

Let $\tau(s,x)$ be the first exit time of the path $\phi_{s,x}$ from the domain K. Since A_- is the domain of attraction of x_- and since K is a compact subset of $A_- \setminus \{x_-\}$, we obtain $\tau(s,x) < \infty$ for all $x \in K$. The function $\tau(s,x)$ is upper semi-continuous with respect to the variables s and x (due to the continuous dependence of $\phi_{s,x}$ on s and x). Hence the maximal value

$$T_1 := \sup_{s\in[0,1],\ x\in K} \tau(s,x)$$

is attained.

Let $T_0 = T_1 + 1$, and consider all functions $\varphi \in C_{0,T_0}$ with values in K. This set of functions is closed with respect to the maximum norm. Since there is no solution of the ordinary differential equation in this set of functions, the functional I^s_{0,T_0} reaches a strictly positive minimum m on this set which is uniform in s. By the additivity of the functional I^s_{0T}, we obtain for $T \geq T_0$ and $\varphi \in C_{0T}$ with values in K

$$I^s_{0T}(\varphi) \geq m \left\lfloor \frac{T}{T_0} \right\rfloor \geq m \left(\frac{T}{T_0} - 1 \right) = c(T - T_0),$$

with $c = \frac{m}{T_0}$. $\square$

Let us recall the subject of interest of this section:

$$\tau_\varrho = \inf \Big\{ t \geq 0 : \ X^\varepsilon_t \in B_\varrho(x_+) \Big\},$$

the hitting time of a small neighborhood of the equilibrium point x_+. First we shall consider upper bounds for the law of this time in some window of length βT^ε, where β is sufficiently small. The important feature of the following statement is that β is independent of s, while the uniformity of the bound again has to be understood in the sense of Remark 5.5.

6.8 Proposition. *For all $\vartheta > 0$ there exist $\beta > 0$, $\eta > 0$ such that for all $s \in [0,1]$*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in B_\eta(x_-)} \mathbb{P}_x\left(sT^\varepsilon \le \tau_\varrho \le (s+\beta)T^\varepsilon\right) \;\le\; \mu - e(s) + \vartheta.$$

This bound holds locally uniformly w.r.t. $\mu \in]\mu_*, e(0)[$ and uniformly w.r.t. $s \in [0,1]$.

Proof. Let M be a compact subset of $]\mu_*, e(0)[$, and fix $\vartheta > 0$. We first introduce some parameter dependent domains the exit times of which will prove to be suitable for estimating the probability that τ_ϱ is in a certain time window.
For this purpose, we define for $\delta > 0$ and $s \in [0,1]$ an open domain

$$D(\delta, s) := \left\{ y \in \mathbb{R}^d : \; V^s(x_-, y) < \mu^* + \frac{1}{1+\delta}, \; \mathrm{dist}(y, A_+) > \delta \right\},$$

and let $D = D(\delta) = \cup_{s\in[0,1]} D(\delta, s)$. Then D is relatively compact in A_-, $\mathrm{dist}(y, A_+) > \delta$ for all $y \in D(\delta)$, and a transition to a ϱ-neighborhood of x_+ certainly requires an exit from $D(\delta)$.
The boundary of $D(\delta)$ consists of two hyper surfaces one of which carries an energy strictly greater than μ^* and thus greater than $e(s)$ for all $s \in [0,1]$. The minimal energy is therefore attained on the other component of distance δ from A_+ which approaches $\chi = \partial A_-$ as $\delta \to 0$. Thus, by the joint continuity of the quasi-potential, we can choose $\delta_0 > 0$ and $\eta > 0$ such that for $\delta \le \delta_0$ and $s \in [0,1]$

$$e(s) = \inf_{z\in\chi} V^s(x_-, z) \ge \inf_{z\in\partial D(\delta)} V^s(x_-, z) \ge \inf_{y\in B_\eta(x_-)} \inf_{z\in\partial D(\delta)} V^s(y, z) \ge e(s) - \frac{\vartheta}{4}. \quad (6.12)$$

Let τ_D be the first exit time of X^ε from D. For $s \in [0,1]$ and $\beta > 0$ we introduce a covering of the interval of interest $[sT^\varepsilon, (s+\beta)T^\varepsilon]$ into $N = N(\beta, \varepsilon, \mu)$ intervals of fixed length ν, i.e. ν is independent of ε, μ, s and β. We will have to assume that ν is sufficiently large which will be made precise later on. Thus we have $N\nu \ge \beta T^\varepsilon$, and we can and do assume that $N \le \beta T^\varepsilon$. For $k \in \mathbb{Z}$, $k \ge -1$, let

$$t_k = t_k(s, \varepsilon, \mu) := sT^\varepsilon + k\nu$$

be the starting points of these intervals. We consider t_{-1} since we need some information about the past in order to ensure the diffusion to start in a neighborhood of the equilibrium x_-. Then for $x \in B_\eta(x_-)$ we get the desired estimation of probabilities of exit windows for τ_ϱ by those with respect to τ_D:

$$\mathbb{P}_x\left(sT^\varepsilon \le \tau_\varrho \le (s+\beta)T^\varepsilon\right) \le \sum_{k=0}^{N} \mathbb{P}_x(t_k \le \tau_D \le t_{k+1}).$$

In a second step we will fix $k \ge 0$ and estimate the probability of a first exit from D during each of the intervals $[t_k, t_{k+1}]$ separately. Here the difficulty is that we don't have any information on the location at time t_k. We therefore condition on whether or not X^ε has entered the neighborhood $B_\eta(x_-)$ in the previous time interval. For that purpose, let

$$\sigma_k := \inf\left\{t \ge t_k \vee 0 : \; X_t^\varepsilon \in B_\eta(x_-)\right\}, \quad k \ge -1.$$

Then for $k \ge 0$

$$\mathbb{P}_x(t_k \le \tau_D \le t_{k+1}) \le \mathbb{P}_x(t_k \le \tau_D \le t_{k+1} | \sigma_{k-1} \le t_k) + \mathbb{P}_x(\tau_D \wedge \sigma_{k-1} \ge t_k). \quad (6.13)$$

In the next step we shall estimate the second term on the right hand side of (6.13). Let $K = K(\delta, \eta) = \overline{D(\delta)} \setminus B_\eta(x_-)$. Then K is compact, and by the Markov property we have

$$\mathbb{P}_x(\tau_D \wedge \sigma_{k-1} \ge t_k) \le \sup_{y \in K} \mathbb{P}_{y, t_{k-1}}(\tau_D \wedge \sigma_1 \ge \nu),$$

where $\mathbb{P}_{y,t}$ is as defined in the previous section. Now we wish to further estimate this exit probability using large deviations methods. The neighborhoods

$$S^{s,\beta}(\varepsilon, \mu) = [sT^\varepsilon - \nu, (s + \nu N(\beta, \varepsilon, \mu))T^\varepsilon]$$

of sT^ε contain each interval $[t_k, t_{k+1}]$, $-1 \le k \le N(\beta, \varepsilon, \mu)$, and they satisfy

$$\limsup_{\varepsilon \to 0} \sup_{\mu \in M, \, s \in [0,1]} \frac{\operatorname{diam}(S^{s,\beta}(\varepsilon, \mu))}{T^\varepsilon} \le \beta.$$

Hence by the uniform LDP of Proposition 5.4, applied to the closed set

$$\Phi_K(\delta, \eta) = \left\{\varphi \in C_{0,\nu} : \; \varphi_t \in K(\delta, \eta) \quad \text{for all} \quad t \in [0, \nu]\right\},$$

we obtain the upper bound

$$\begin{aligned}
&\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in K, \, \mu \in M, \, k \le N} \mathbb{P}_{y, t_{k-1}}(\tau_D \wedge \sigma_1 \ge \nu) \\
&\le \limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in K, \, \mu \in M, \, t \in S^{s,\beta}(\varepsilon, \mu)} \mathbb{P}_{y,t}(X^\varepsilon \in \Phi_K(\delta, \eta)) \qquad (6.14) \\
&\le - \inf_{y \in K} \inf_{\varphi \in \Phi_K(\delta, \eta)^{\gamma_0(\beta)}} I_{0,\nu}^s(\varphi),
\end{aligned}$$

where $\gamma_0(\beta) = \beta\delta(\Phi_K(\delta,\eta))$ is the 'blowup-factor' induced by the diameter β. Since $\gamma_0(\beta) \to 0$ as $\beta \to 0$, we can find $\beta_0 > 0$ such that for $\beta \le \beta_0$

$$\Phi_K(\delta,\eta)^{\gamma_0(\beta)} \subset \Phi_K\Big(\tfrac{\delta}{2},\tfrac{\eta}{2}\Big),$$

which amounts to saying that, instead of blowing up the set of paths, we consider the slightly enlarged domain $K(\frac{\delta}{2},\frac{\eta}{2})$. Thus

$$-\inf_{y\in K}\ \inf_{\varphi\in\Phi_K(\delta,\eta)^{\gamma_0(\beta)}} I^s_{0,\nu}(\varphi) \le -\inf_{y\in K}\ \inf_{\varphi\in\Phi_K(\frac{\delta}{2},\frac{\eta}{2})} I^s_{0,\nu}(\varphi).$$

By Lemma 6.7 the latter expression, and therefore the r.h.s. of (6.14), approaches $-\infty$ as $\nu \to \infty$, uniformly w.r.t. $s \in [0,1]$. So the second term in the decomposition (6.13) of $\mathbb{P}_x(t_k \le \tau_D \le t_{k+1})$ can be neglected since it becomes exponentially small of any desired order by choosing ν suitably large.

In the next and most difficult step, we treat the first term on the r.h.s. of (6.13). It is given by the probability that, while X^ε is in $B_\eta(x_-)$ at time σ_{k-1}, it exits within a time interval of length $t_{k+1} - \sigma_{k-1} \le 2\nu$. Hence by the strong Markov property

$$\mathbb{P}_x(t_k \le \tau_D \le t_{k+1}|\sigma_{k-1} \le t_k) \le \sup_{t_{k-1}\le t\le t_k,\ y\in B_\eta(x_-)} \mathbb{P}_{y,t}(\tau_D \le 2\nu).$$

Applying the uniform LDP to the closed set

$$F_D(\delta) := \{\varphi \in C_{0,2\nu} :\ \varphi_0 \in D(\delta),\ \varphi_{t_0} \notin D(\delta) \text{ for some } t_0 \le 2\nu\},$$

yields the upper bound

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in B_\eta(x_-),\ \mu\in M,\ t\in S^{s,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,t}(\tau_D \le 2\nu) \le -\inf_{y\in B_\eta(x_-)}\ \inf_{\varphi\in F_D(\delta)^{\gamma_0(\beta)}} I^s_{0,2\nu}(\varphi),$$

where $\gamma_0(\beta) = 2\beta\delta(F_D(\delta))$. By the same reasoning as before we can replace the blow-up of the path sets $F_D(\delta)$ by an enlargement of the domain $D(\delta)$. We find $\beta_1 > 0$ such that for $\beta \le \beta_1$

$$\begin{aligned} -\inf_{y\in B_\eta(x_-)}\ \inf_{\varphi\in F_D(\delta)^{\gamma_0(\beta)}} I^s_{0,2\nu}(\varphi) &\le -\inf_{y\in B_\eta(x_-)}\ \inf_{\varphi\in F_D(\frac{\delta}{2})} I^s_{0,2\nu}(\varphi) \\ &\le -\inf_{y\in B_\eta(x_-)}\ \inf_{z\in\partial D(\frac{\delta}{2})} V^s(y,z). \end{aligned}$$

Now we apply (6.12) and recall the uniformity of the LDP w.r.t. s. We find $\varepsilon_0 > 0$ such that we have for $\varepsilon \le \varepsilon_0$, $s \in [0,1]$, $\mu \in M$ and $\beta \le \beta_1$

$$\begin{aligned} \varepsilon \log \sup_{y\in B_\eta(x_-),\ t\in S^{s,\beta}(\varepsilon,\mu)} \mathbb{P}_{y,t}(\tau_D \le 2\nu) &\le -\inf_{y\in B_\eta(x_-)}\ \inf_{z\in\partial D(\frac{\delta}{2})} V^s(y,z) + \frac{\vartheta}{4} \\ &\le -e(s) + \frac{\vartheta}{2}. \end{aligned} \tag{6.15}$$

We finally summarize our findings. We conclude that there exists $\varepsilon_1 > 0$ such that for $\varepsilon \le \varepsilon_1$, $\mu \in M$ and $s \in [0,1]$ we have

$$\begin{aligned}
&\varepsilon \log \sup_{x \in B_\eta(x_-)} \mathbb{P}_x(sT^\varepsilon \le \tau_\varrho \le (s+\beta)T^\varepsilon) \\
&\le \varepsilon \log \Big\{ \sum_{k=0}^{N(\beta,\varepsilon,\mu)} \sup_{x \in B_\eta(x_-)} \mathbb{P}_x \left(t_k \le \tau_D \le t_{k+1} \middle| \sigma_{k-1} \le t_k \right) \Big\} + \frac{\vartheta}{4} \\
&\le \varepsilon \log \Big\{ \beta T^\varepsilon \exp\Big(-\frac{1}{\varepsilon}\Big[e(s) - \frac{\vartheta}{2} \Big] \Big) \Big\} + \frac{\vartheta}{4} \\
&= \varepsilon \log \beta + \mu - e(s) + \frac{3}{4}\vartheta \\
&\le \mu - e(s) + \vartheta.
\end{aligned}$$

This completes the proof. $\square$

6.9 Remark. If we stay away from $s = 0$, in the statement of Proposition 6.8 the radius of the starting domain $B_\eta(x_-)$ can be chosen independently of the parameter ϑ. It may then be brought into the following somewhat different form.

6.10 Proposition. *Let L and M be compact subsets of $]0,1]$ resp. $]\mu_*, e(0)[$. Let $\eta > 0$ be small enough such that $B_\eta(x_-)$ belongs to the domain*

$$\{y \in \mathbb{R}^d : \ V^s(x_-, y) < \mu^* \quad \textit{for all} \quad s \in L\}.$$

Then for each $\vartheta > 0$ there exists some $\beta > 0$ such that

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x \in B_\eta(x_-)} \mathbb{P}(sT^\varepsilon \le \tau_\varrho \le (s+\beta)T^\varepsilon) \ \le \ \mu - e(s) + \vartheta,$$

uniformly w.r.t $s \in L$ and $\mu \in M$.

Proof. To prove Proposition 6.10, one has to slightly modify the preceding proof. Instead of just η one has to choose two different parameters: η_0 for the definition of the starting domain D and some η_1 for the description of the location of the diffusion at time t_k, i.e. for the definition of the stopping times σ_k. $\square$

In the following Proposition, we derive the upper bound for the asymptotic law of transition times, corresponding to the lower bound obtained in Proposition 6.6.

6.11 Proposition. *Let $\mu < e(0)$, and recall from (6.5) the definition $a_\mu = \inf\{t \ge 0 : \ e(t) \le \mu\}$. There exist $\gamma > 0$ and $h_0 > 0$ such that for all $h \le h_0$*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x \in B_\gamma(x_-)} \mathbb{P}_x \left(\tau_\varrho \le (a_\mu - h)T^\varepsilon \right) \ \le \ \mu - e(a_\mu - h). \tag{6.16}$$

This bound is uniform w.r.t. μ on compact subsets of $]\mu_*, e(0)[$.

Proof. Let M be a compact subset of $]\mu_*, e(0)[$. To choose h_0, we use our assumptions on the geometry of the energy function e. Recall Assumption 6.2 according to which e is strictly monotonous in the open intervals between the extrema $]\mu_*, \mu^*[$. It implies that e is monotonically decreasing on the interval $[a_{e(0)}, a_\mu]$ for any $\mu \in M$. By choice of M, we further have $a_{e(0)} < \inf_{\mu\in M} a_\mu$. Now choose h_0 such that

$$\inf_{\mu\in M} a_\mu - h_0 > a_{e(0)}.$$

Then we have for $h \leq h_0$

$$\inf_{\mu\in M} a_\mu - h > 0, \tag{6.17}$$

$$e(0) > \sup_{\mu\in M, h\leq h_0} e(a_\mu - h), \tag{6.18}$$

$$e(s) \geq e(a_\mu - h) \quad \text{for all} \quad s \leq a_\mu - h. \tag{6.19}$$

To see (6.19), note that for $0 \leq s \leq a_{e(0)}$, by definition of $a_{e(0)}$, the inequality $e(s) \geq e(0) > e(a_\mu - h)$ holds, while for $a_{e(0)} \leq s \leq a_\mu - h$ by monotonicity $e(s) \geq e(a_\mu - h)$.

Next fix $h \leq h_0$. For $\mu \in M$, let $\Lambda_0 = \Lambda_0(\mu) = 0$, and $\Lambda_1(\mu) \leq \inf_{\mu\in M}(a_\mu - h)T^\varepsilon$. For $N \in \mathbb{N}^*$ we set $\Lambda_i(\mu) = \Lambda_1 + \frac{i-1}{N-1}((a_\mu - h)T^\varepsilon - \Lambda_1)$, $2 \leq i \leq N$, thus splitting the time interval $[0, (a_\mu - h)T^\varepsilon]$ into the N intervals $[\Lambda_i(\mu), \Lambda_{i+1}(\mu)]$, $0 \leq i \leq N-1$. Then for $\gamma > 0$, $x \in B_\gamma(x_-)$

$$\mathbb{P}_x(\tau_\varrho \leq (a_\mu - h)T^\varepsilon) \leq \sum_{i=0}^{N-1} \mathbb{P}_x(\tau_\varrho \in [\Lambda_i(\mu), \Lambda_{i+1}(\mu)]),$$

which implies

$$\begin{aligned}\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in B_\gamma(x_-)} &\mathbb{P}_x(\tau_\varrho \leq (a_\mu - h)T^\varepsilon) \\ &\leq \max_{0\leq i\leq N-1} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in B_\gamma(x_-)} \mathbb{P}_x(\tau_\varrho \in [\Lambda_i(\mu), \Lambda_{i+1}(\mu)]).\end{aligned}$$

Fix $\vartheta > 0$ such that for $h \leq h_0, \mu \in M$ we have $e(0) \geq e(a_\mu - h) + \vartheta$. This is guaranteed by (6.18). We shall show that

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x\in B_\gamma(x_-)} \mathbb{P}_x(\tau_\varrho \in [\Lambda_i(\mu), \Lambda_{i+1}(\mu)]) \leq \mu - e(a_\mu - h) + \vartheta$$

uniformly in $0 \leq i \leq N-1$ and $\mu \in M$.

Let us treat the estimation of the first term separately from the others. In fact, by Proposition 6.8, setting $s = 0$, $\beta = \Lambda_1/T^\varepsilon$, we may choose Λ, $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that for $\Lambda_1 \leq \Lambda T^\varepsilon, \varepsilon \leq \varepsilon_0, \gamma \leq \gamma_0, \mu \in M$ the inequality

$$\varepsilon \log \sup_{x\in B_\gamma(x_-)} \mathbb{P}_x(\tau_\varrho \in [\Lambda_0(\mu), \Lambda_1(\mu)]) \leq \mu - e(0) + \vartheta$$

holds. Now we use the inequality $e(0) \geq e(a_\mu - h) + \vartheta$, valid for all $\mu \in M$. Hence there exists $\Lambda > 0$, $\varepsilon_0 > 0$ and $\gamma_0 > 0$ such that for $\Lambda_1 \leq \Lambda T^\varepsilon, \varepsilon \leq \varepsilon_0, \gamma \leq \gamma_0, \mu \in M$

$$\varepsilon \log \sup_{x \in B_\gamma(x_-)} \mathbb{P}_x(\tau_\varrho \in [\Lambda_0(\mu), \Lambda_1(\mu)]) \leq \mu - e(a_\mu - h).$$

Let us next estimate the contributions for the intervals $[\Lambda_i(\mu), \Lambda_{i+1}(\mu)]$ with $i \geq 1$. We use Proposition 6.8, this time with $s = \Lambda_i(\mu)/T^\varepsilon$, $\beta = \frac{1}{N-1} \sup_{\mu \in M} a_\mu$. By the definition of a_μ, we get $e(s) > e(a_\mu)$ for all $s < a_\mu$. By (6.19), we have $e(s) = e(\Lambda_i(\mu)/T^\varepsilon) \geq e(a_\mu - h)$. By Remark 6.9,

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x \in B_\gamma(x_-)} \mathbb{P}_x(\tau_\varrho \in [\Lambda_i(\mu), \Lambda_{i+1}(\mu)]) \leq \mu - e(a_\mu - h) + \vartheta$$

uniformly w.r.t $1 \leq i \leq N$ and $\mu \in M$. Letting ϑ tend to 0, which implies that N tends to infinity and Λ_1 tends to zero, we obtain the desired upper bound for the exponential exit rate. $\square$

Chapter 7

Stochastic resonance

Given the results of the previous chapter on the asymptotics of exit times which are uniform in the scale parameter μ, we are now in a position to consider the problem of finding a satisfactory probabilistic notion of stochastic resonance that does not suffer from the lack of robustness defect of physical notions such as spectral power amplification. We continue to study the SDE

$$dX_t^\varepsilon = b\Big(\frac{t}{T^\varepsilon}, X_t^\varepsilon\Big)dt + \sqrt{\varepsilon}\, dW_t, \quad t \geq 0,\ X_0^\varepsilon = x_0 \in \mathbb{R}^d$$

introduced before, thereby recalling that the drift term b satisfies the local Lipschitz conditions (5.8) and (5.7) in space and time, as well as the growth condition (5.9). Moreover, b is assumed to be one-periodic in time such that T^ε is the period of the deterministic input of the randomly perturbed dynamical system described by X^ε.

In typical applications $b = -\nabla_x U$ is given by the (spatial) gradient of some time periodic double-well potential U (see Pavlyukevich [38]). The potential possesses at all times two local minima well separated by a barrier. The depth of the wells and the roles of being the deep and shallow one change periodically. The diffusion X^ε then roughly describes the motion of a Brownian particle of intensity ε in a double-well landscape. Its attempts to get close to the energetically most favorable deep position in the landscape makes it move along random trajectories which exhibit randomly periodic hopping between the wells. The average time the trajectories need to leave a potential well of depth $\frac{v}{2}$ being given by the Kramers-Eyring law $T^\varepsilon = \exp\left\{\frac{v}{\varepsilon}\right\}$ motivates our choice of time scales $T^\varepsilon = \exp\left\{\frac{\mu}{\varepsilon}\right\}$ and also our convention to measure time scales in their corresponding energy units μ.

The problem of stochastic resonance consists of characterizing the *optimal tuning* of the noise, i.e. the best relation between the noise amplitude ε and the input period T^ε – or, in our units the energy parameter μ – of the deterministic system which makes the diffusion trajectories look as periodic as possible. Of course, the optimality criterion must be based upon a *quality measure* for periodicity in random trajectories.

In this chapter we shall develop a measure of quality based on the transition probabilities investigated in Chapter 6. With respect to this measure we identify a resonance energy $\mu_0(\varepsilon)$ for which the diffusion trajectories follow the periodic forcing of the system at intensity ε in an optimal way. We shall study the problem in the general situation introduced in Chapter 6, which includes the double-well potential gradient case as an important example, and draws its intuition from it. The deterministic system

$$\dot{\varphi}_s(t) = b(s, \varphi_s(t)), \quad t \geq 0,$$

has to satisfy Assumption 6.1, i.e. it possesses two well separated domains of attraction the common boundary of which is time invariant.

In the first section we shall describe the *resonance interval*, i.e. the set of scale parameters μ for which trivial behavior, i.e. either constant or continuously jumping trajectories, are excluded. The second section shows that a quality measure of periodic tuning is given by the exponential rate at which the first transition from one domain of attraction to the other one happens within a fixed time window around $a_\mu T^\varepsilon$. This quality measure is robust, as demonstrated in the last section: in the small noise limit the diffusion and its reduced model, a Markov chain jumping between the domains of attraction reduced to the equilibrium points, display the same resonance pattern.

7.1 Resonance interval

According to Freidlin [18], quasi-periodic hopping behavior of the trajectories of our diffusion in the small noise limit of course requires that the energies required to leave the domains of attraction of the two equilibria switch their order periodically: if $e_\pm$ denotes the energy needed to leave $A_\pm$, then e_+ needs to be bigger than e_- during part of one period, and vice versa for the rest. We assume that $e_\pm$ both satisfy Assumption 6.2 and associate to each of these functions the transition time

$$a_\mu^\pm(s) = \inf\Big\{t \geq s : \ e_\pm(t) \leq \mu\Big\}.$$

The scale parameters μ for which relevant behavior of the system is expected clearly belong to the intervals

$$I_i \ = \ \Big]\inf_{t\geq 0} e_i(t), \sup_{t\geq 0} e_i(t)\Big[, \quad i \in \{-, +\}.$$

Our aim being the observation of periodicity, we have to make sure that the process can travel back and forth between the domains of attraction on the time scales considered, but not instantaneously. So, on the one hand, in these scales it should not *get stuck* in one of the domains. On the other hand, they should not allow for *chaotic behavior*, i.e. immediate re-bouncing after leaving a domain has to be avoided.

To make these conditions mathematically precise, recall that transitions from A_i to A_{-i} become possible as soon as the energy e_i needed to exit from A_i falls below μ, which represents the available energy. Not to get stuck in one of $A_\pm$, we therefore have to guarantee

$$\mu > \max_{i=\pm} \inf_{t\geq 0} e_i(t).$$

To avoid immediate re-bouncing, we have to assure that the diffusion cannot leave A_{-i} at the moment it reaches it, coming from A_i. Suppose we consider the dynamics after time $s \geq 0$, and the diffusion is near i at that time. Its first transition to A_{-i} occurs at time $a^i_\mu(s)T^\varepsilon$ where $a^i_\mu(s)$ is the first time in the original scale at which e_i falls below μ after s. Provided $e_{-i}(a^i_\mu(s))$ is bigger than μ, it stays there for at least a little while. This is equivalent to saying that for all $s \geq 0$ there exists $\delta > 0$ such that on $[a^i_\mu(s), a^i_\mu(s)+\delta]$ we have $\mu < e_{-i}$. Since by definition for t shortly after $a^i_\mu(s)$, we always have $e_i(t) \leq \mu$, our condition may be paraphrased by: for all $s \geq 0$ there exists $\delta > 0$ such that on $[a^i_\mu(s), a^i_\mu(s) + \delta]$ we have $\mu < \max_{i=\pm} e_i$. This in turn is more elegantly expressed by

$$\mu < \inf_{t\geq 0} \max_{i=\pm} e_i(t).$$

Our search for a set of scales μ for which the diffusion exhibits non-trivial transition behavior may be summarized in the following definition. The interval

$$I_R \;=\; \Big]\max_{i=\pm} \inf_{t\geq 0} e_i(t), \inf_{t\geq 0} \max_{i=\pm} e_i(t)\Big[$$

is called *resonance interval* (see Figure 7.1).

In this interval, for small ε, we have to look for an optimal energy scale $\mu_0(\varepsilon)$. See [23] and [24] for the definition of the corresponding interval in the one-dimensional case and in the case of two state Markov chains. In Freidlin's [18] terms, stochastic resonance in the sense of quasi-deterministic periodic motion is given if the parameter μ exceeds the lower boundary of our resonance interval.

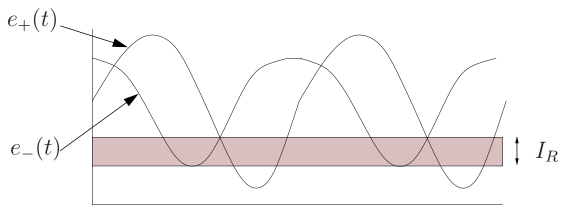

Figure 7.1: Resonance interval

Let us briefly comment on the potential gradient case. Assume $b(t,x) = -\nabla_x U(t,x)$, $t \geq 0$, $x \in \mathbb{R}^d$, where U is some time periodic double-well potential with time invariant local minima $x_\pm$ and separatrix. Then A_- and A_+ represent the two wells of the potential, χ the separatrix. The energy $e_\pm$ is, in fact, the energy some Brownian particle needs to cross χ. Lemma 3.2 due to Freidlin and Wentzell [19] gives the link between this energy and the depth of the well. If $D_\pm(t) = \inf_{y\in\chi} U(t,y) - U(t,x_\pm)$ denote the depths of the wells, then $e_\pm(t) = 2D_\pm(t)$ for all $t \geq 0$. Hence the resonance interval is given by

$$I_R = \Big]\max_{i=\pm 1}\inf_{t\geq 0} 2D_i(t),\ \inf_{t\geq 0}\max_{i=\pm 1} 2D_i(t)\Big[.$$

7.2 Transition rates as quality measure

Let us now explain in detail our measure of quality designed to give a concept of optimal tuning which, as opposed to physical measures (see Pavlyukevich [38] and Chapter 4), is robust for model reduction to Markov chains just retaining the jump dynamics between the meta-stable equilibria of the diffusion. We shall use a notion that is based just on this rough transition mechanism. In fact, generalizing an approach for two state Markov chain models (see [24]), we measure the quality of tuning by computing for varying energy parameters μ the probability that, starting in x_i, the diffusion is transferred to x_{-i} within the time window $[(a_\mu^i(0)-h)T^\varepsilon, (a_\mu^i(0)+h)T^\varepsilon]$ of width $2hT^\varepsilon$. To find the *stochastic resonance point* for large T^ε (small ε) we have to maximize this measure of quality w.r.t. $\mu \in I_R$. The probability for transition within this window will be approximated using the estimates of Chapter 6. Uniformity of convergence to the exponential rates will enable us to maximize in μ for fixed small ε.

Let us now make these ideas precise. To make sure that the transition window makes sense at least for small h, we have to suppose that $a_\mu^i > 0$ for $i = \pm 1$ and $\mu \in I_R$. This is guaranteed if

$$e_i(0) > \inf_{t\geq 0}\max_{j=\pm} e_j(t), \quad i = \pm.$$

If this is not granted from the beginning, it suffices to start the diffusion a little later. For ϱ small enough such that $B_\varrho(x_\pm) \subset A_\pm$, we call

$$\mathcal{M}(\varepsilon,\mu,\varrho) = \min_{i=\pm}\ \sup_{x\in B_\varrho(x_i)} \mathbb{P}_x\Big(\tau_\varrho^{-i} \in [(a_\mu^i - h)T^\varepsilon, (a_\mu^i + h)T^\varepsilon]\Big), \quad \varepsilon > 0, \mu \in I_R, \quad (7.1)$$

transition probability for a time window of width h. Here

$$\tau_\varrho^i = \inf\Big\{t \geq 0 : \ X_t^\varepsilon \in B_\varrho(x_i)\Big\},$$

and a_μ^i abbreviates $a_\mu^i(0)$, the transition time at zero. We are ready to state our main result on the asymptotic law of transition time windows. This is an obvious consequence of Theorem 6.3.

7.1 Theorem. *Let M be a compact subset of I_R, $h_0 > 0$ and ϱ be given according to Theorem 6.3. Then for all $h \leq h_0$*

$$\lim_{\varepsilon\to 0}\varepsilon\log\Big(1 - \mathcal{M}(\varepsilon,\mu,\varrho)\Big) = \max_{i=\pm}\Big\{\mu - e_i(a_\mu^i - h)\Big\} \quad (7.2)$$

uniformly w.r.t. $\mu \in M$.

It is clear that for h small the eventually existing global minimizer $\mu_R(h)$ of

$$I_R \ni \mu \mapsto \max_{i=\pm 1} \left\{ \mu - e_i(a_\mu^i - h) \right\}$$

is a good candidate for our resonance point. But it still depends on h. To get rid of this dependence, we shall consider the limit of $\mu_R(h)$ as $h \to 0$.

7.2 Definition. Suppose that

$$I_R \ni \mu \mapsto \max_{i=\pm} \left\{ \mu - e_i(a_\mu^i - h) \right\}$$

possesses a global minimum $\mu_R(h)$. Suppose further that

$$\mu_R = \lim_{h \to 0} \mu_R(h)$$

exists in I_R. We call μ_R the *stochastic resonance point* of the diffusion X^ε.

Let us now illustrate this resonance notion in a situation in which the energy functions are related by a phase lag $\alpha \in]0,1[$, i.e. $e_-(t) = e_+(t+\alpha)$ for all $t \geq 0$. We shall show that in this case the stochastic resonance point exists if one of the energy functions, and thus both, has a unique point of maximal decrease on the interval where it is strictly decreasing.

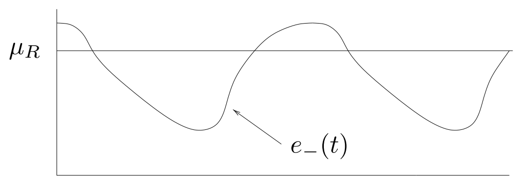

Figure 7.2: Point of maximal decrease

7.3 Theorem. *Suppose that e_- is twice continuously differentiable and has its global maximum at t_1, and its global minimum at t_2, where $t_1 < t_2$. Suppose further that there is a unique point $t_1 < s < t_2$ such that $e_-|_{]t_1,s[}$ is strictly concave, and $e_-|_{]s,t_2[}$ is strictly convex. Then $\mu_R = e_-(s)$ is the stochastic resonance point.*

Proof. As a consequence of the phase lag of the energy functions,

$$\max_{i=\pm} \left\{ \mu - e_i(a_\mu^i - h) \right\} = \mu - e_-(a_\mu^- - h).$$

Write $a_\mu = a_\mu^-$, and recall that on the interval of decrease of e_- we have $a_\mu = e_-^{-1}(\mu)$. Therefore, the differentiability assumption yields

$$1 = e'_-(a_\mu - h) \cdot a'_\mu = e'_-(a_\mu - h) \frac{1}{e'_-(a_\mu)}.$$

Our hypotheses concerning convexity and concavity of e_- essentially means that $e''_-(s) = 0$, and $e''_-|_{]t_1,s[} < 0, e''_-|_{]s,t_2[} > 0$, which may be stated alternatively by saying that $\mu \mapsto e'_-(a_\mu)$ has a local maximum at $a_\mu = s$. Hence for h small there exists a unique point $a_\mu(h)$ such that

$$e'_-(a_\mu(h) - h) = e'_-(a_\mu(h))$$

and

$$\lim_{h\to 0} a_\mu(h) = s.$$

To show that $a_\mu(h)$ corresponds to a minimum of the function

$$\mu \mapsto \mu - e_-(a_\mu - h),$$

we take the second derivative of this function at $a_\mu(h)$, which is given by

$$\frac{e'_-(a_\mu(h) - h)e''_-(a_\mu(h)) - e''_-(a_\mu(h) - h)e'_-(a_\mu(h))}{e'_-(a_\mu(h))}.$$

But $e'_-(a_\mu(h)), e'_-(a_\mu(h) - h) < 0$, whereas $e''_-(a_\mu(h) - h) > 0, e''_-(a_\mu(h)) < 0$. This clearly implies that $a_\mu(h)$ corresponds to a minimum of the function. But by definition, as $h \to 0$, $a_\mu(h) \to s$. Therefore, finally, $e_-(s)$ is the stochastic resonance point. □

7.3 The robustness of stochastic resonance

In the small noise limit $\varepsilon \to 0$, it seems reasonable to assume that the periodicity properties of the diffusion trajectories caused by the periodic forcing the drift term exhibits, are essentially captured by a simpler, reduced stochastic process: a continuous time Markov chain which just jumps between two states representing the equilibria in the two domains of attraction. Jump rates correspond to the transition mechanism of the diffusion. This is just the reduction idea ubiquitous in the physics literature, and explained for example in McNamara, Wiesenfeld [33]. We shall now show that in the small noise limit both models, diffusion and Markov chain, produce the same resonance picture, if quality of periodic tuning is measured by transition rates.

To describe the reduced model, let $e_\pm$ be the energy functions corresponding to transitions from $A_\mp$ to $A_\pm$ as before. Assume a phase locking of the two functions according to the previous section, i.e. assume that $e_-(t) = e_+(t+\alpha), t \geq 0$, with phase shift $\alpha \in]0,1[$. Let us consider a continuous time Markov chain $\{Y_t^\varepsilon, t \geq 0\}$ taking values in the state space $S = \{-,+\}$ with initial data $Y_0^\varepsilon = -$. Suppose the infinitesimal generator is given by

$$G(t) = \begin{pmatrix} -\phi(\frac{t}{T^\varepsilon}) & \phi(\frac{t}{T^\varepsilon}) \\ \tilde{\phi}(\frac{t}{T^\varepsilon}) & -\tilde{\phi}(\frac{t}{T^\varepsilon}) \end{pmatrix},$$

where $\tilde{\phi}(t) = \phi(t + \alpha)$, $t \geq 0$, and ϕ is a 1-periodic function describing a rate which just produces the transition dynamics of the diffusion between the equilibria $\pm$, i.e.

$$\phi(t) = \exp\Big\{ - \frac{e_+(t)}{\varepsilon} \Big\}, \quad t \geq 0. \tag{7.3}$$

Note that by choice of ϕ,

$$\tilde{\phi}(t) = \exp\Big\{ - \frac{e_-(t)}{\varepsilon} \Big\}, \quad t \geq 0. \tag{7.4}$$

This reduced model was studied intensely by Herrmann and Imkeller [24]. As a quality measure for the Markov chain Y^ε, they investigate the corresponding analogue of (7.1) for the Markov chain Y^ε.

Let σ_i denote the first transition time of Y^ε. For $t \geq 0$ its probability density is given by

$$\begin{aligned} p(t) &= \phi(t) \exp\Big\{ - \int_0^t \phi(s) ds \Big\}, \quad \text{if} \quad i = -, \\ q(t) &= \tilde{\phi}(t) \exp\Big\{ - \int_0^t \tilde{\phi}(s) ds \Big\}, \quad \text{if} \quad i = +, \end{aligned} \tag{7.5}$$

see [24], Section 2. Equation (7.5) can be used to obtain results on exponential rates of the transition times σ_i if starting from $-i$, $i = \pm$. We summarize them and apply them to the following measure of quality of periodic tuning

$$\mathcal{N}(\varepsilon, \mu) = \min_{i=\pm} \mathbb{P}_i \Big(\sigma_{-i} \in [(a^i_\mu - h)T^\varepsilon, (a^i_\mu + h)T^\varepsilon] \Big), \quad \varepsilon > 0, \, \mu \in I_R, \tag{7.6}$$

which is called *transition probability for a time window of width h* for the Markov chain.

Here is the asymptotic result obtained from a slight modification of Theorem 3 of [24], which consists of allowing for more general energy functions than the sinusoidal ones used there. In fact, the sinusoidal shape does not enter the arguments of their proof at all, so it holds literally.

7.4 Theorem. *Let M be a compact subset of I_R and $h_0 < a^-_\mu$. Then for $0 < h \leq h_0$*

$$\lim_{\varepsilon \to 0} \varepsilon \ln(1 - \mathcal{N}(\varepsilon, \mu)) = \max_{i=\pm 1} \Big\{ \mu - e_-(a^i_\mu - h) \Big\} \tag{7.7}$$

uniformly w.r.t. $\mu \in M$.

It is clear from Theorem 7.4 that the Markov chain Y^ε and the diffusion process X^ε have exactly the same resonance behavior. Of course, we may define the *stochastic resonance point* for Y^ε just as we did for X^ε. So the following final robustness result holds true.

7.5 Theorem. *The resonance points of X^ε with time periodic drift b and of Y^ε with exponential transition rate functions $e_\pm$ coincide.*

Part III

Large deviations and the exit problem for self-stabilizing diffusions

Chapter 8

Interacting particle systems and self-stabilizing diffusions

8.1 Introduction

The notion of stochastic resonance presented in the previous chapter describes the resonance behavior of one isolated particle whose dynamics is governed by a bistable time-periodic drift term and perturbed by Gaussian noise of small amplitude. Despite the fact that the optimal tuning based on transition times is satisfactory from a mathematical point of view – principally since the quality measure has proved to be robust – the model could be generalized and improved in various directions.

One aspect that is of particular importance in many physical and biological systems consists in coupling. This is notably true for microscopic applications, e.g. resonance patterns observed in neural activity or in the behavior of electromagnetic systems. There one usually cannot observe one isolated unit as the object of interest, instead one faces a complex dynamics of many small objects that interact with each other in a way that makes up the intrinsic properties of the system. In such systems one observes a whole ensemble of objects or particles. Each particle's dynamics depends on the configuration of the entire particle system, which results in a global coupling of the whole particle ensemble.

It is well known from empirical expertise that such coupled systems do indeed exhibit stochastic resonance behavior, and one expects that, due to a collective response resulting from coupling, the resonance effect will be more pronounced than for an isolated particle. In a simplest situation, a two-dimensional system was investigated by Neiman and Schimansky-Geier [35] (see also [21], Section VII.B). They consider two coupled overdamped bistable particles described by the SDE

$$\begin{aligned} dx_t &= [\alpha_1 x_t - x_t^3 - \gamma(x_t - y_t) + A\cos(\Omega t)]dt + \sqrt{\varepsilon} dW_t^1 \\ dy_t &= [\alpha_2 y_t - y_t^3 - \gamma(y_t - x_t) + A\cos(\Omega t)]dt + \sqrt{\varepsilon} dW_t^2 \end{aligned} \tag{8.1}$$

with independent Brownian motions W^1 and W^2. Here each particle is driven by the quartic bistable potential $x \mapsto \frac{1}{4}x^4 - \frac{\alpha_i}{2}x^2$ and subject to the same periodic and noisy perturbations. But contrary to the classical isolated case, a linear coupling term whose amplitude is quantified by the coupling constant γ comes into play. In [35] two main effects have been observed for the system (8.1). Firstly, for a fixed coupling constant, the signal-to-noise ratio of the sum $x_t + y_t$ exhibits a maximum as a function of ε. Secondly, the signal-to-noise ratio is analyzed as a function of the coupling strength γ, while the noise intensity remains fixed. The case $\gamma = 0$ corresponds to two independent systems. As γ increases, the signal-to-noise ratio goes through a maximum and finally approaches some finite asymptotic value. These observations support the intuitive idea that the 'collective' response of the coupled system (8.1) is stronger than that of the two uncoupled systems.

More generally, one may investigate higher dimensional coupled systems with many degrees of freedom in the presence of periodic forcing, i.e.

$$dx_t^n = \Big[- U'(x_t^n) - \frac{\gamma}{N}\sum_{i=1}^{N}(x_t^n - x_t^i) + A\cos(\Omega t)\Big]dt + \sqrt{\varepsilon} dW_t^n, \qquad n = 1, \dots N, \tag{8.2}$$

where U is again a bistable potential, for example $U(x) = \frac{1}{4}x^4 - \frac{1}{2}x^2$.

Without periodic forcing ($A = 0$), this is the Curie-Weiss model of ferromagnetism. The model has also been used to describe the dynamics of muscle contraction (see [34] and the references therein).

In the presence of periodic forcing, i.e. $A > 0$, the spectral amplification of (8.2) strongly increases with the coupling strength γ and exhibits a peak as γ reaches a critical level that corresponds to a noise intensity at which the system undergoes a phase transition. The stochastic resonance effect is again enhanced by the feedback that results from coupling ([34]). Similar effects have been observed in coupled neuron models ([21], section VII.B).

As in the uncoupled case, these 'physical' notions for measuring stochastic resonance effects (i.e. SPA and SNR) will not be robust and will exhibit the same disadvantages as without coupling. It seems therefore natural to extend the probabilistic quality measure investigated in the previous part to the coupled system (8.2). In order to treat such high dimensional systems, one usually takes the so-called *hydrodynamic limit* $N \to \infty$, which leads to the *mean-field approximation* of (8.2). The rationale behind this is that the x^i satisfy a law of large numbers, i.e. $\frac{1}{N}\sum_{i=1}^N x_t^i \to \mathbb{E}[x_t^1]$, which leads to the limiting equation

$$dx_t = \Big[- U'(x_t) - \gamma(x_t - \mathbb{E}[x_t]) + A\cos(\Omega t)\Big] dt + \sqrt{\varepsilon} dW_t. \tag{8.3}$$

This low-dimensional model describes the approximate behavior of one particle belonging to the system (8.2). It is frequently studied in place of the original high-dimensional system, both from a numerical and an analytical point of view. Its resonance behavior can be investigated via simulations.

In order to give a probabilistic description of periodicity for the trajectories of (8.3), much of the classical work still needs to be done. The quality measure discussed in the previous chapter relies essentially on Kramers' law for time homogeneous diffusions. Due to the additional drift term $\gamma(x_t - \mathbb{E}[x_t])$ in (8.3) that originates in coupling, such a law for the diffusion (8.3) without periodic forcing ($A = 0$) cannot be deduced from the classical theory in an obvious way.

In mathematical terms (8.3) is a *self-stabilizing* diffusion: the trajectories of (8.3) are attracted by their average position. In the forthcoming chapters, we shall investigate a general class of self-stabilizing diffusions, given as solutions of the stochastic differential equation

$$dX_t^\varepsilon = V(X_t^\varepsilon)\, dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)\, du_t^\varepsilon(x)\, dt + \sqrt{\varepsilon} dW_t. \tag{8.4}$$

In this equation, V denotes a vector field on $\mathbb{R}^d$, which we think of as representing a potential gradient, and W is a Brownian motion. The second drift term involving the process' own law u_t^ε introduces self-stabilization. The distance between the particle's instantaneous position X_t^ε and a fixed point x in state space is weighed by means of a so-called *interaction function* Φ and integrated in x against the law u_t^ε of X_t^ε itself. This effective additional drift can be seen as a measure for the average attractive force exerted on the particle by an independent copy of itself through the attraction potential Φ. In effect, this forcing makes the diffusion inertial and stabilizes its motion in certain regions of the state space.

In this part of the thesis, we shall investigate the large deviations behavior of (8.4) and extend the classical Kramers' law to this class of diffusions. This is only a first step towards the mathematically challenging problem of analyzing the stochastic resonance behavior of self-stabilizing, periodically perturbed systems such as (8.3).

The subsequent chapters are organized as follows. In the next section we introduce the mathematical setting for interacting systems such as (8.2), and explain the connection to self-stabilizing diffusions (McKean-Vlasov limit). In Section 8.3 the existence and uniqueness of strong solutions to the self-stabilizing diffusion equation (8.4) is established. This is required for the techniques employed in Chapter 9, where a large deviations principle for (8.4) is derived. In Chapter 10 we prove the announced extension of Kramers' law, which is illustrated by examples and a discussion of the one-dimensional case in Chapter 11.

8.2 Interacting diffusions and the McKean-Vlasov limit

In this section we shall introduce the mathematical setting for coupled systems like (8.2) and their corresponding limit (8.3). A very instructive introduction to the topic may be found in [45].

The dynamics of an interacting particle system like (8.2) is described by a system of *weakly interacting diffusions*. On a probability space $(\Omega, \mathscr{F}, \mathbb{P})$, we consider the system of d-dimensional SDEs

$$\begin{aligned} dX_t^{i,N} &= \frac{1}{N}\sum_{j=1}^{N} B(X_t^{i,N}, X_t^{j,N})\,dt + \sigma\,dW_t^i, \qquad i = 1,\dots,N, \\ X_0^{i,N} &= x_0^i, \end{aligned} \tag{8.5}$$

where the W^i are independent Brownian motions. (8.5) is a coupled system of N diffusions in d-dimensional Euclidean space that interact weakly via their empirical measures

$$u_t^N(A,\omega) = \frac{1}{N}\sum_{j=1}^{N} \delta_{X_t^{j,N}(\omega)}(A), \quad A \in \mathscr{B}(\mathbb{R}^d). \tag{8.6}$$

Namely, if we denote $B[x,\nu] = \int B(x,y)\nu(dy)$ for a measure ν on $\mathbb{R}^d$ and $x \in \mathbb{R}^d$, then the drift term of (8.5) may be rewritten as

$$\frac{1}{N}\sum_{j=1}^{N} B(x, X_t^{j,N}) = B[x, u_t^N].$$

Assume that (8.5) admits a unique strong solution. The drift function $B(x,y)$ describes the interaction between two particles located at x and y in state space. For each particle, this effect is averaged over the entire population of particles, resulting in a global coupling of the particle ensemble. In typical applications the drift term may be split into two components,

$$B(x,y) = V(x) - \Phi(x,y), \tag{8.7}$$

with an external force V that describes the intrinsic geometry that governs each particle's own dynamics and an interaction part Φ which renders the particle population's influence on each individual particle. Such an assumption will also take effect in subsequent chapters, where V mimics the geometric structure of a potential gradient, i.e. $V \approx -\nabla U$ for a potential U on $\mathbb{R}^d$. In a more general setting, the diffusion matrix σ may also depend on the state variable, and one may consider an additional interaction through the diffusive part of the SDE, but we shall rule this out in our treatment.

The *empirical process* $u^N = (u_t^N)_{t\geq 0}$ is the key object to gain analytical and numerical access to the high-dimensional system (8.5). It is a measure-valued Markov process taking values in the space of probability measures on $\mathbb{R}^d$. As stated below, under mild conditions on B, the random probability measures $u_t^N = u_t^N(\omega, \cdot)$ satisfy a law of large numbers (the McKean-Vlasov limit) and converge in the large particle limit $N \to \infty$ to a deterministic probability measure-valued process $u = (u_t)_{t\geq 0}$. Replacing u_t^N in (8.5) by its limiting counterpart gives rise to a new system described by the equations

$$\begin{aligned} dX_t^i &= B[X_t^i, u_t]dt + \sigma dW_t^i, \qquad i \in \mathbb{N}, \\ \mathcal{L}(X_t^i) &= u_t, \qquad t \geq 0. \end{aligned} \tag{8.8}$$

Here $\mathcal{L}(X_t^i) = u_t$ means that X_t^i is distributed according to u_t, i.e. the coupling of particles is replaced by a coupling of each diffusion to its own law. According to Sznitman [45], Theorem 1.4, if B is bounded and Lipschitz, then

$$\mathbb{E}\left[\sup_{0\leq t\leq T} \left\|X_t^{i,N} - X_t^i\right\|\right] \leq \frac{c_T}{\sqrt{N}}$$

for some constant c_T, i.e. $X_t^{i,N}$ tends to X_t^i for each i, so that (8.8) yields a low-dimensional approximation of (8.5). Indeed, due to the independence of the Brownian motions, the X^i in (8.8) are independent, and it suffices to study only one particle of the system (8.8).

The following theorem gives the formal result about the convergence of the empirical process. It is a special case of Theorem 6.2 in [30], where a second coupling through the diffusion coefficient is allowed for.

8.1 Theorem (McKean-Vlasov limit)**.** *Assume that the initial conditions of* (8.5) *satisfy* $x_0^i \in L^4(\Omega, \mathscr{F}, \mathbb{P})$ *for each* $i \in \mathbb{N}$*. Suppose that the drift term* $B : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R}^d$ *is split according to* (8.7)*, and let* $\Phi_1, \Phi_2 : \mathbb{R}^d \to \mathbb{R}_+$ *such that for all* $x, y \in \mathbb{R}^d$

$$\|\Phi(x,y)\| \leq \Phi_1(x) + \Phi_2(y),$$

and suppose that there exists $K > 0$ *such that the following conditions are satisfied:*

(i) $V(x)$ *and* $x \mapsto \Phi(x,y)$ *are locally Lipschitz for each* $y \in \mathbb{R}^d$*,*

(ii) $y \mapsto \Phi(x,y)$ *is globally Lipschitz, uniformly w.r.t.* $x \in \mathbb{R}^d$*,*

(iii) $\langle x, V(x)\rangle + \|x\|\,\Phi_1(x) \le K(1+\|x\|^2)$ *for all* $x \in \mathbb{R}^d$,

(iv) $\Phi_2(y) \le K(1+\|y\|)$ *for all* $y \in \mathbb{R}^d$,

(v) *there exist* $r \ge 0$ *such that* $\|V(x)\| + \Phi_1(x) \le K(1+\|x\|^r)$ *for all* $x \in \mathbb{R}^d$.

If there exists a probability measure u_0 *on* $\mathbb{R}^d$ *such that* $u_0^N \to u_0$ *in the weak sense and*

$$\sup_{N\ge 1} \mathbb{E}\left[\int \|x\|^p \; u_0^N(dx,\cdot)\right] < \infty \quad \textit{for some} \quad p \ge \max(4, 2r),$$

then the probability-measure valued process u^N *defined by* (8.6) *converges weakly to the law* $u = (u_t)_{t\ge 0}$ *of the unique strong solution of* (8.8) *with initial law* u_0.

The assertion of this theorem contains in particular that (8.8) admits a unique strong solution. There is a possible tradeoff between integrability conditions of initial conditions and the growth conditions of the coefficients (see [30]).
The weak convergence $u^N \to u$, i.e. the convergence of the laws $\mathcal{L}(u^N)$ to the law δ_u of the deterministic measure-valued process is a rather abstract formulation. Since the empirical process u^N is a probability measure on $C(\mathbb{R}_+, \mathbb{R}^d)$, i.e. an element of the Polish space $\mathcal{P}(C(\mathbb{R}_+, \mathbb{R}^d))$, the weak convergence of $\mathcal{L}(u^N)$ takes place in the sense of weak convergence of measures in $\mathcal{P}(\mathcal{P}(C(\mathbb{R}_+, \mathbb{R}^d)))$.
In case the initial laws of (8.5) are exchangeable (i.e. for each N the law of $(x_0^1, \dots, x_0^N)$ is invariant against permutations), the weak convergence of $\mathcal{L}(u^N)$ is equivalent to

$$(X^{1,N}, \dots, X^{m,N}) \xrightarrow[N\to\infty]{} (X^1, \dots, X^m) \qquad \text{weakly in } \; \mathcal{P}(C(\mathbb{R}_+, \mathbb{R}^d))$$

for all $m \in \mathbb{N}$, i.e. $\mathcal{L}(X^{1,N}, \dots, X^{m,N}) \to u^{\otimes m}$ (see [30]). This holds in particular if the initial laws are independent identically distributed. In that case one speaks of *propagation of chaos*, which means the following. The initial laws are i.i.d (i.e. 'chaotic'), but after a positive amount of time, due to coupling, the random variables $(X_t^{1,N}, \dots, X_t^{m,N})$ are dependent. However, as $N \to \infty$, the processes become asymptotically independent again: the chaos propagates.

Depending on structural properties of the drift function B, the system (8.5) exhibits qualitatively different behavior. For instance, one may think of rejecting particles in dimension one. If the force of mutual rejection is sufficiently strong, intuition tells us that particles will not be able to exchange positions and will keep their initial order over the course of time, so that no propagation of chaos shall occur.
In the sequel, we shall confine ourselves to the attractive case. By a *self-stabilizing diffusion* we mean a solution of (8.8) where the interaction force Φ in the decomposition (8.7) depends solely on the particle distance, i.e.

$$\Phi(x, y) = \Phi(x - y),$$

and Φ increases with the distance of x and y. Under broader growth conditions than in Theorem 8.1, we shall study self-stabilizing diffusions in their own right, and examine their large deviations and exit behavior in the small noise limit.

The study of interacting systems like (8.5) was initiated by McKean [32]. Various generalizations and variations have been investigated since then. A survey about the general setting for interaction (under global Lipschitz and boundedness assumptions) may be found in [45]. Here the existence of the McKean-Vlasov limit, propagation of chaos as well as the link to Burgers' equation are established. Large deviations of the particle system from the McKean-Vlasov limit were investigated by Dawson and Gärtner [14]. Further results about the McKean-Vlasov limit, in particular its existence under broader assumptions on the coefficients, were obtained by Gärtner [22] and Léonard [30]. See also [11] and [31].
A strictly local form of interaction was investigated by Stroock and Varadhan in simplifying its functional description to a Dirac measure [44]. Oelschläger studies the particular case where interaction is represented by the derivative of the Dirac measure at zero [37]. Funaki addresses existence and uniqueness for the martingale problem associated with self-stabilizing diffusions [20]. Scheutzow [42] studies uniqueness of (8.8) in the degenerate case $\sigma = 0$, and without making structural assumptions that constitute self-stabilization.
The behavior of self-stabilizing diffusions, in particular the convergence to invariant measures, was studied by various authors under different assumptions on the structure of the interaction, see e.g. [46], [47], [4] and [5].

8.3 Existence and uniqueness of self-stabilizing diffusions

In this section, we shall discuss the existence of solutions of the diffusion equation (8.4), that is

$$dX_t^\varepsilon = V(X_t^\varepsilon)\,dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)\,du_t^\varepsilon(x)\,dt + \sqrt{\varepsilon} dW_t. \tag{8.9}$$

The existence of unique strong solutions to this equation is essential for the techniques employed in the subsequent chapter in order to study the large deviations behavior. It is non-trivial in our situation, since the solution process' own law appears in the equation. The interesting interaction term $\int \Phi(X_t^\varepsilon - x)\,du_t^\varepsilon(x)$ also adds a considerable amount of complexity to the mathematical treatment. It depends on $u_t^\varepsilon = \mathbb{P} \circ (X_t^\varepsilon)^{-1}$, thus classical existence and uniqueness results on SDEs as well as the classical results on large deviations as stated in Chapters 1 and 2 are not directly applicable. Consequently, the question of existence and uniqueness of solutions for equation (8.9) is an integral part in any discussion of the self-stabilizing diffusion's behavior.

We follow Benachour et al. [4] to design a recursive procedure in order to prove the existence of the *interaction drift* $b(t,x) = \int \Phi(x-y)\, du_t^\varepsilon(y)$, the second drift component of (8.9). More precisely, we shall construct a locally Lipschitz drift term $b(t,x)$ such that the classical SDE

$$dX_t^\varepsilon = V(X_t^\varepsilon)\,dt - b(t,X_t^\varepsilon)\,dt + \sqrt{\varepsilon} dW_t,\ t \geq 0, \tag{8.10}$$

admits a unique strong solution, which satisfies the additional condition

$$b(t,x) = \int_{\mathbb{R}^d} \Phi(x-y)\,du_t^\varepsilon(y) = \mathbb{E}\left\{\Phi(x - X_t^\varepsilon)\right\}. \tag{8.11}$$

In (8.10) W is a standard d-dimensional Brownian motion, and $V : \mathbb{R}^d \to \mathbb{R}^d$ mimics the geometrical structure of a potential gradient. Existence and uniqueness for equation (8.9) will be understood in the sense that (8.10) and (8.11) hold with a unique b and a pathwise unique process X^ε. For locally Lipschitz interaction functions of at most polynomial growth, Benachour et al. [4] have proved the existence of strong solutions in the one-dimensional situation, and in the absence of the vector field V. Since V forces the diffusion to spend even more time in bounded sets due to its dissipativity formulated below, it imposes no complications concerning questions of existence and uniqueness. Our arguments rely on a modification of their construction.

Besides some Lipschitz type regularity conditions on the coefficients, we make assumptions concerning the geometry of V and Φ which render the system (8.9) dissipative in a suitable sense. All necessary conditions are summarized in the following assumption.

8.2 Assumption.

i) The coefficients V and Φ are locally Lipschitz, i.e. for $R > 0$ there exists $K_R > 0$ s.t.

$$\|V(x) - V(y)\| + \|\Phi(x) - \Phi(y)\| \ \leq\ K_R \|x-y\| \tag{8.12}$$

for $x, y \in B_R(0) = \{z \in \mathbb{R}^d :\ \|z\| < R\}$.

ii) The interaction function Φ is rotationally invariant, i.e. there exists an increasing function $\phi : [0,\infty) \to [0,\infty)$ with $\phi(0) = 0$ such that

$$\Phi(x) = \frac{x}{\|x\|}\phi(\|x\|), \quad x \in \mathbb{R}^d. \tag{8.13}$$

iii) Φ grows at most polynomially: there exist $K > 0$ and $r \in \mathbb{N}$ such that

$$\|\Phi(x) - \Phi(y)\| \ \leq\ \|x-y\| \left(K + \|x\|^r + \|y\|^r\right), \quad x, y \in \mathbb{R}^d. \tag{8.14}$$

iv) *V is continuously differentiable. Let $DV(x)$ denote the Jacobian of V. We assume that there exist $K_V > 0$ and $R_0 > 0$ such that*

$$\langle h, DV(x)h \rangle \leq -K_V \tag{8.15}$$

for $h \in \mathbb{R}^d$ s.t. $\|h\| = 1$ and $x \in \mathbb{R}^d$ s.t. $\|x\| \geq R_0$.

The conditions that make our diffusion dissipative are (8.13) and (8.15). (8.13) means that the interaction is essentially not more complicated than in the one-dimensional situation and has some important implications for the geometry of the drift component $\mathbb{E}\left[\Phi(x - X_t^\varepsilon)\right]$ originating from self-interaction, namely that it points back to the origin. The same holds true for V due to (8.15). In the gradient case $V = -\nabla U$, $-DV$ is the Hessian of U, and (8.15) means that its eigenvalues are uniformly bounded from below (w.r.t. x) on neighborhoods of ∞. (8.14) is just a convenient way to combine polynomial growth and the local Lipschitz assumption in one condition. In the following two lemmas we summarize a few simple consequences of these assumptions.

8.3 Lemma. *There exist constants $K, \eta, R_1 > 0$ such that the following holds true:*

a) *For all $x, y \in \mathbb{R}^d$*

$$\Big\langle x - y, V(x) - V(y) \Big\rangle \leq K \|x - y\|^2 . \tag{8.16}$$

b) *For $x, y \in \mathbb{R}^d$ such that $\|x - y\| \geq R_1$*

$$\Big\langle x - y, V(x) - V(y) \Big\rangle \leq -\eta \|x - y\|^2 . \tag{8.17}$$

c) *For $x \in \mathbb{R}^d$ with $\|x\| \geq R_1$*

$$\Big\langle x, V(x) \Big\rangle \leq -\eta \|x\|^2 . \tag{8.18}$$

Proof. Note first that, by continuity of DV, there exists $K > 0$ such that

$$\langle h, DV(x)h \rangle \leq K$$

holds for *all* x and all h of norm 1. Moreover, for $x, y \in \mathbb{R}^d$, $x \neq y$, we have

$$\frac{V(x) - V(y)}{\|x - y\|} = \int_0^1 DV(y + t(x - y)) \frac{x - y}{\|x - y\|} \, dt,$$

and therefore

$$\left\langle \frac{x - y}{\|x - y\|}, \frac{V(x) - V(y)}{\|x - y\|} \right\rangle = \int_0^1 \Big\langle h, DV(y + t \|x - y\| h) h \Big\rangle \, dt, \tag{8.19}$$

where $h := \frac{x-y}{\|x-y\|}$. Since the integrand is bounded by K, this proves a).
For b), observe that the proportion of the line connecting x and y that lies inside $B_{R_0}(0)$ is at most $\frac{2R_0}{\|x-y\|}$. Hence

$$\left\langle \frac{x-y}{\|x-y\|}, \frac{V(x)-V(y)}{\|x-y\|} \right\rangle \leq K\frac{2R_0}{\|x-y\|} - K_V\Big(1 - \frac{2R_0}{\|x-y\|}\Big),$$

which yields b).
c) is shown in a similar way. Let $x \in \mathbb{R}^d$ with $\|x\| > R_0$, and set $y := R_0\frac{x}{\|x\|}$. Then the same argument shows the sharper bound

$$-K_V \geq \left\langle \frac{x-y}{\|x-y\|}, \frac{V(x)-V(y)}{\|x-y\|} \right\rangle = \left\langle \frac{x}{\|x\|}, \frac{V(x)-V(y)}{\|x\|-R_0} \right\rangle,$$

since the line connecting x and y does not intersect $B_{R_0}(0)$. Hence

$$\langle x, V(x)\rangle \leq -K_V\,\|x\|\,(\|x\|-R_0) + \|x\|\,\|V(y)\|\,,$$

which shows that (8.18) is satisfied if we set $R_1 = \max\{2R_0, 4\sup_{\|y\|=R_0}\frac{\|V(y)\|}{K_V}\}$ and $\eta = \frac{K_V}{4}$. □

8.4 Lemma. *For all $x, y, z \in \mathbb{R}^d$ we have*

a) $\|\Phi(x-y)\| \leq 2K + \Big(K + 2^{r+1}\Big)\Big(\,\|x\|^{r+1} + \|y\|^{r+1}\,\Big),$

b) $\|\Phi(x-z) - \Phi(y-z)\| \;\leq\; \|x-y\|\,\Big[K + 2^r\Big(\,\|x\|^r + \|y\|^r + 2\,\|z\|^r\,\Big)\Big],$

c) $\|\Phi(x-y) - \Phi(x-z)\| \;\leq\; K_1\,\|y-z\|\,\Big(1 + \|x\|^r\Big)\Big(1 + \|y\|^r + \|z\|^r\Big),$
where $K_1 = \max(K, 2^{r+1})$.

d) For all $x, y \in \mathbb{R}^d$ and $n \in \mathbb{N}$

$$\Big\langle x\,\|x\|^n - y\,\|y\|^n\,, \Phi(x-y)\Big\rangle \geq 0. \tag{8.20}$$

Proof. By (8.14) and since $\Phi(0) = 0$ we have

$$\begin{aligned}
\|\Phi(x-y)\| \;&\leq\; \|x-y\|\,\Big(K + \|x-y\|^r\Big) \\
&\leq\; K\Big(\,\|x\| + \|y\|\,\Big) + 2^{r+1}\Big(\,\|x\|^{r+1} + \|y\|^{r+1}\Big) \\
&\leq\; K\Big(2 + \|x\|^{r+1} + \|y\|^{r+1}\,\Big) + 2^{r+1}\Big(\,\|x\|^{r+1} + \|y\|^{r+1}\Big) \\
&=\; 2K + \Big(K + 2^{r+1}\Big)\Big(\,\|x\|^{r+1} + \|y\|^{r+1}\Big),
\end{aligned}$$

i.e. a) is proved. For b), we use (8.14) again to see that

$$\begin{aligned} \|\Phi(x-z)-\Phi(y-z)\| &\leq \|x-y\| \Big(K+\|x-z\|^r+\|y-z\|^r\Big) \\ &\leq \|x-y\| \Big[K+2^r\Big(\|x\|^r+\|y\|^r+2\|z\|^r\Big)\Big]. \end{aligned}$$

Property c) follows from $\Phi(-x) = -\Phi(x)$ by further exploiting b) as follows. We have

$$\|\Phi(x-y)-\Phi(x-z)\| \leq \|x-y\| \Big[K+2^{r+1}\Big(\|x\|^r+\|y\|^r+\|z\|^r\Big)\Big],$$

which obviously yields c). Finally, d) follows from a simple calculation and (8.13). Obviously, (8.20) is equivalent to $\langle x\|x\|^n - y\|y\|^n, x-y\rangle \geq 0$. But this is an immediate consequence of the Schwarz inequality. □

Let us now return to the construction of a solution to (8.9), i.e. a solution to the pair (8.10) and (8.11). The crucial property of these coupled equations is that the drift b depends on (the law of) X^ε and therefore also on V, ε and the initial condition x_0. This means that a solution of (8.10) and (8.11) consists of a pair (X^ε, b), a continuous stochastic process X^ε and a drift term b, that satisfies these two equations. For convenience, we shall drop the ε-dependence of X^ε in the sequel.

Our construction of such a pair (X, b) shall focus on the existence of the interaction drift b. It will be constructed as a fixed point in an appropriate function space such that the corresponding solution of (8.10) fulfills (8.11). Let us first derive some properties of b that follow from (8.11).

8.5 Lemma. *Let $T > 0$, and let $(X_t)_{0\leq t\leq T}$ be a stochastic process such that*

$$\sup_{0\leq t\leq T} \mathbb{E}\Big[\|X_t\|^{r+1}\Big] < \infty.$$

Then $b(t,x) = \mathbb{E}\Big[\Phi(x-X_t)\Big]$ has the following properties:

a) b is locally Lipschitz w.r.t. $x \in \mathbb{R}^d$, and the Lipschitz constant is independent of $t \in [0,T]$.

b) $\Big\langle x-y, b(t,x)-b(t,y)\Big\rangle \geq 0$ for all $x,y \in \mathbb{R}^d$, $t \in [0,T]$.

c) b grows polynomially of order $r+1$.

Proof. Note first that $y \mapsto \Phi(x-y)$ grows polynomially of order $r+1$ by Lemma 8.4 a), so that b is well-defined. Moreover, we have

$$\|b(t,x)\| \leq \mathbb{E}\Big[\|\Phi(x-X_t)\|\Big] \leq 2K+\Big(K+2^{r+1}\Big)\Big(\|x\|^{r+1}+\mathbb{E}\Big[\|X_t\|^{r+1}\Big]\Big),$$

which proves c). For a) observe that, by Lemma 8.4 b), we have for $z \in \mathbb{R}^d$, $x, y \in B_R(0)$

$$\|\Phi(x-z) - \Phi(y-z)\| \;\leq\; \|x-y\| \left[K + 2^{r+1}\Big(R^r + \|z\|^r\Big)\right].$$

Hence

$$\begin{aligned}\|b(t,x) - b(t,y)\| \;&\leq\; \mathbb{E}\Big[\, \|\Phi(x-X_t) - \Phi(y-X_t)\| \,\Big] \\ &\leq\; \|x-y\| \left[K + 2^{r+1}\Big(R^r + \mathbb{E}\Big[\, \|X_t\|^r \,\Big]\Big)\right]\end{aligned}$$

for $x, y \in B_R(0)$. Since $\sup\limits_{0 \leq t \leq T} \mathbb{E}\Big[\, \|X_t\|^{r+1} \,\Big] < \infty$, this implies a).

In order to prove b), fix $t \in [0,T]$, and let $\mu = \mathbb{P} \circ X_t^{-1}$. Then

$$\langle x-y, b(t,x) - b(t,y)\rangle = \int \left\langle x-y, \frac{x-u}{\|x-u\|}\phi(\|x-u\|) - \frac{y-u}{\|y-u\|}\phi(\|y-u\|)\right\rangle \mu(du).$$

The integrand is non-negative. Indeed, it equals

$$\begin{aligned}&\|x-u\|\,\phi(\|x-u\|) + \|y-u\|\,\phi(\|y-u\|) \\ &\quad - \left\langle y-u, \frac{x-u}{\|x-u\|}\phi(\|x-u\|)\right\rangle - \left\langle x-u, \frac{y-u}{\|y-u\|}\phi(\|y-u\|)\right\rangle \\ \geq\;& \|x-u\|\,\phi(\|x-u\|) + \|y-u\|\,\phi(\|y-u\|) \\ &\quad - \|y-u\|\,\phi(\|x-u\|) - \|x-u\|\,\phi(\|y-u\|) \\ =\;& (\|x-u\| - \|y-u\|)(\phi(\|x-u\|) - \phi(\|y-u\|)),\end{aligned}$$

which is non-negative since ϕ is increasing, so b) is established. □

In the light of the preceding lemma it is reasonable to define a space of functions that satisfy the above stated conditions, and to look for a candidate for the drift function in this space. Let $T > 0$, and for a continuous function $b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d$ define

$$\|b\|_T := \sup_{t \in [0,T]} \sup_{x \in \mathbb{R}^d} \frac{\|b(t,x)\|}{1 + \|x\|^{2q}}, \tag{8.21}$$

where $q \in \mathbb{N}$ is a fixed constant such that $2q > r$, the order of the polynomial growth of the interaction function Φ. Furthermore, let

$$\overline{\Lambda}_T := \left\{ b : [0,T] \times \mathbb{R}^d \to \mathbb{R}^d \,\middle|\, \|b\|_T < \infty,\ x \mapsto b(t,x) \text{ is locally Lipschitz, uniformly w.r.t. } t \right\}. \tag{8.22}$$

Lemma 8.5 shows that, besides being an element of $\overline{\Lambda}_T$, the drift of (8.10) must satisfy the dissipativity condition

$$\Big\langle x-y, b(t,x) - b(t,y)\Big\rangle \geq 0, \quad x, y \in \mathbb{R}^d. \tag{8.23}$$

Therefore, we define

$$\Lambda_T := \Big\{ b \in \overline{\Lambda}_T : b \text{ satisfies } (8.23) \Big\}. \tag{8.24}$$

It is obvious that $\|\cdot\|_T$ is indeed a norm on the vector space $\overline{\Lambda}_T$. The subset Λ_T will be the object of interest for our construction of the interaction drift in what follows, i.e. we shall construct the interaction drift as an element of Λ_T for a proper choice of the time horizon T.

Once we have constructed the drift, the diffusion X will simply be given as the unique strong solution of (8.10) due to the classical result of Corollary 2.2 about strong solvability of SDEs. It ensures the existence of a unique strong solution to (8.10) for a *given* drift b. Indeed, as an immediate consequence of (8.18) and (8.23), the drift $\beta(t, x) = V(x) - b(t, x)$ satisfies the assumptions of Corollary 2.2 for any $b \in \Lambda_T$.

To construct a solution of (8.9), we proceed in two steps. In the first and technically most demanding step, we construct a drift on a small time interval $[0, T]$. We shall define an operator Γ such that (8.11) translates into a fixed point property for this operator. To ensure the existence of a fixed point, one needs contraction properties of Γ which shall turn out to depend on the time horizon T. This way we obtain a drift defined on $[0, T]$ such that the associated solution X exists up to time T. In a second step, we show that this solution's moments are uniformly bounded w.r.t. time, which guarantees non-explosion and allows us to extend X to the whole time axis.

To carry out this program, we start by comparing diffusions with different drift terms.

8.6 Lemma. *For $b^1, b^2 \in \Lambda_T$ consider the associated diffusions*

$$dY_t = V(Y_t)\,dt - b^1(t, Y_t)\,dt + \sqrt{\varepsilon} dW_t$$

and

$$dZ_t = V(Z_t)\,dt - b^2(t, Z_t)\,dt + \sqrt{\varepsilon} dW_t,$$

and assume $Y_0 = Z_0$. Then for $t \le T$

$$\|Y_t - Z_t\| \le e^{KT} \left\| b^1 - b^2 \right\|_T \int_0^t \Big(1 + \|Z_s\|^{2q}\Big) ds.$$

Proof. Since $Y - Z$ is governed by a (pathwise) ODE, we have

$$\begin{aligned}
\|Y_t - Z_t\| = &\int_0^t \Big\langle \frac{Y_s - Z_s}{\|Y_s - Z_s\|}, V(Y_s) - V(Z_s) \Big\rangle ds \\
&- \int_0^t \Big\langle \frac{Y_s - Z_s}{\|Y_s - Z_s\|}, b^1(s, Y_s) - b^1(s, Z_s) \Big\rangle ds \\
&+ \int_0^t \Big\langle \frac{Y_s - Z_s}{\|Y_s - Z_s\|}, b^2(s, Z_s) - b^1(s, Z_s) \Big\rangle \, ds.
\end{aligned}$$

The second integral in this decomposition is positive by definition of Λ_T, so it can be neglected. Furthermore, the first integral is bounded by $K\int_0^t \|Y_s - Z_s\|\, ds$ due to the dissipativity condition (8.16) on V. The last integral is bounded by

$$\int_0^t \left\|b^2(s,Z_s) - b^1(s,Z_s)\right\| ds \leq \left\|b^1 - b^2\right\|_T \int_0^t (1 + \|Z_s\|^{2q})\, ds.$$

Combining these estimates yields

$$\|Y_t - Z_t\| \leq K\int_0^t \|Y_s - Z_s\|\, ds + \left\|b^1 - b^2\right\|_T \int_0^t (1 + \|Z_s\|^{2q})\, ds.$$

Now an application of Gronwall's lemma completes the proof. □

The liberty of choice for the drift terms in Lemma 8.6 allows us to get bounds on Y and its moments by making a particular one for Z. We consider the special case of a linear drift term $b(t,x) = \lambda x$.

8.7 Lemma. *Let $\lambda \geq K$, and let Z be the solution of*

$$dZ_t = V(Z_t)\, dt - \lambda Z_t\, dt + \sqrt{\varepsilon}\, dW_t.$$

Furthermore, assume that $\mathbb{E}(\|Z_0\|^{2m}) < \infty$ for some $m \in \mathbb{N}$, $m \geq 1$.
Then for all $t \geq 0$

$$\mathbb{E}\left[\, \|Z_t\|^{2m} \right] \;\leq\; 2mt\, \|V(0)\|\, R_1^{2m-1} \exp\left\{ \frac{\varepsilon(dm+m-1)t}{R_1^2} \right\}, \qquad \text{if } Z_0 = 0 \text{ a.s.},$$

and

$$\begin{aligned} \mathbb{E}\left[\, \|Z_t\|^{2m} \right] \;\leq\; & \mathbb{E}\left[\, \|Z_0\|^{2m} \right] \exp\left\{ \frac{\varepsilon(dm+m-1)t}{\left(\mathbb{E}\left[\, \|Z_0\|^{2m}\right]\right)^{\frac{1}{m}}} \right\} \\ & + 2mt\, \|V(0)\|\, R_1^{2m-1} \exp\left\{ \frac{\varepsilon(dm+m-1)t}{R_1^2} \right\}, \end{aligned}$$

otherwise.

Proof. By Itô's formula we have for $n \geq 2$

$$\begin{aligned} \|Z_t\|^n \;=\; & \|Z_0\|^n + M_t^n + n\int_0^t \|Z_s\|^{n-2} \left\langle Z_s, V(Z_s) \right\rangle - \lambda \|Z_s\|^n\, ds \\ & + \frac{\varepsilon}{2}(dn+n-2) \int_0^t \|Z_s\|^{n-2}\, ds, \end{aligned} \tag{8.25}$$

where M^n is the local martingale $M_t^n = n\sqrt{\varepsilon}\int_0^t \langle Z_s \|Z_s\|^{n-2}, dW_s\rangle$.
Since $\langle x, V(x)\rangle \leq -\eta \|x\|^2$ for $\|x\| > R_1$ according to (8.18), the first integrand

of (8.25) is negative if $\|Z_s\| > R_1$. If $\|Z_s\| \leq R_1$, we use the global estimate $\langle x, V(x)\rangle \leq K\|x\|^2 + \|V(0)\|\,\|x\|$, which follows from (8.16). We deduce that, since $\lambda \geq K$,

$$\|Z_s\|^{n-2}\Big\langle Z_s, V(Z_s)\Big\rangle - \lambda\|Z_s\|^n \leq (K-\lambda)\|Z_s\|^n + \|V(0)\|\,\|Z_s\|^{n-1} \leq \|V(0)\|\,R_1^{n-1}.$$

Thus,

$$\|Z_t\|^n \leq \|Z_0\|^n + M_t^n + n\|V(0)\|\,tR_1^{n-1} + \frac{\varepsilon}{2}(dn+n-2)\int_0^t \|Z_s\|^{n-2}\,ds. \tag{8.26}$$

Using a localization argument and monotone convergence yields

$$\mathbb{E}\Big[\,\|Z_t\|^n\,\Big] \leq \mathbb{E}\Big[\,\|Z_0\|^n\,\Big] + n\|V(0)\|\,tR_1^{n-1} + \frac{\varepsilon}{2}(dn+n-2)\int_0^t \mathbb{E}\Big[\,\|Z_s\|^{n-2}\,\Big]\,ds. \tag{8.27}$$

We claim that this implies

$$\mathbb{E}\Big[\,\|Z_t\|^{2m}\,\Big] \;\leq\; \sum_{j=0}^m \mathbb{E}\Big[\,\|Z_0\|^{2(m-j)}\,\Big]\frac{(\alpha_m t)^j}{j!} + 2m\frac{\|V(0)\|}{\alpha_m}R_1^{2m+1}\sum_{j=1}^m \frac{(\alpha_m t)^j}{R_1^{2j}j!} \tag{8.28}$$

for all $m \in \mathbb{N}$, $m \geq 1$, where $\alpha_m = \varepsilon(dm+m-1)$. Indeed, for $m=1$ this is evidently true by (8.27). The general case follows by induction. Assume (8.28) holds true for $m-1$. Then by (8.27)

$$\begin{aligned}
&\mathbb{E}\Big[\,\|Z_t\|^{2m}\,\Big] \\
&\qquad\leq \mathbb{E}\Big[\,\|Z_0\|^{2m}\,\Big] + 2m\|V(0)\|\,tR_1^{2m-1} \\
&\qquad\quad + \alpha_m\int_0^t \Bigg\{\sum_{j=1}^m \mathbb{E}\Big[\,\|Z_0\|^{2(m-j)}\,\Big]\frac{(\alpha_{m-1}s)^{j-1}}{(j-1)!} \\
&\qquad\qquad\qquad + 2(m-1)\frac{\|V(0)\|}{\alpha_{m-1}}R_1^{2m-1}\sum_{j=2}^m \frac{(\alpha_{m-1}s)^{j-1}}{R_1^{2(j-1)}(j-1)!}\Bigg\}\,ds \\
&\qquad\leq \mathbb{E}\Big[\,\|Z_0\|^{2m}\,\Big] + 2m\|V(0)\|\,tR_1^{2m-1} \\
&\qquad\quad + \sum_{j=1}^m \alpha_m\,\mathbb{E}\Big[\,\|Z_0\|^{2(m-j)}\,\Big]\frac{\alpha_{m-1}^{j-1}\,t^j}{j!} + 2m\|V(0)\|\,R_1^{2m-1}\sum_{j=2}^m \alpha_m\frac{\alpha_{m-1}^{j-2}\,t^j}{R_1^{2(j-1)}j!} \\
&\qquad\leq 2m\|V(0)\|\,tR_1^{2m-1} + \sum_{j=0}^m \mathbb{E}\Big[\,\|Z_0\|^{2(m-j)}\,\Big]\frac{\alpha_m^j\,t^j}{j!} \\
&\qquad\quad + 2m\|V(0)\|\,R_1^{2m+1}\sum_{j=2}^m \frac{\alpha_m^{j-1}\,t^j}{R_1^{2j}j!} \\
&\qquad= \sum_{j=0}^m \mathbb{E}\Big[\,\|Z_0\|^{2(m-j)}\,\Big]\frac{\alpha_m^j\,t^j}{j!} + 2m\|V(0)\|\,R_1^{2m+1}\sum_{j=1}^m \frac{\alpha_m^{j-1}\,t^j}{R_1^{2j}j!},
\end{aligned}$$

and so (8.28) is established. Since $\mathbb{E}\left[\|Z_0\|^{2(m-j)}\right] \le \left(\mathbb{E}\left[\|Z_0\|^{2m}\right]\right)^{1-\frac{j}{m}}$ for $j \le m$, we may exploit (8.28) further to conclude that

$$\begin{aligned}
&\mathbb{E}\left[\|Z_t\|^{2m}\right] \\
&\le \mathbb{E}\left[\|Z_0\|^{2m}\right]\sum_{j=0}^{m}\frac{\alpha_m^j\, t^j}{j!\left(\mathbb{E}\left[\|Z_0\|^{2m}\right]\right)^{\frac{j}{m}}} + 2mt\,\|V(0)\|\,R_1^{2m-1}\sum_{j=1}^{m}\frac{\alpha_m^{j-1}\,t^{j-1}}{R_1^{2j-2}j!} \\
&\le \mathbb{E}\left[\|Z_0\|^{2m}\right]\exp\left\{\frac{\alpha_m t}{\left(\mathbb{E}\left[\|Z_0\|^{2m}\right]\right)^{\frac{1}{m}}}\right\} + 2mt\,\|V(0)\|\,R_1^{2m-1}\exp\left\{\frac{\alpha_m t}{R_1^2}\right\},
\end{aligned}$$

which is the announced bound if we identify the first term as zero in case $Z_0 = 0$. □

Let us define the mapping Γ on Λ_T that will be a contraction under suitable conditions. For $b \in \Lambda_T$, denote by $X^{(b)}$ the solution of

$$dX_t = V(X_t)\,dt - b(t,X_t)\,dt + \sqrt{\varepsilon}dW_t, \tag{8.29}$$

and let $\Gamma b(t,x) := \mathbb{E}\left[\Phi\left(x - X_t^{(b)}\right)\right]$. By combining the two previous lemmas, we obtain the following a priori bound on the moments of $X^{(b)}$.

8.8 Lemma. *Let $n \in \mathbb{N}$. Assume that the initial datum of (8.29) satisfies*

$$\mathbb{E}\left[\left\|X_0^{(b)}\right\|^{2qn}\right] < \infty.$$

For each $T > 0$ there exists $k = k(n,T) > 0$ such that for all $b \in \Lambda_T$

$$\sup_{0\le t\le T}\mathbb{E}\left[\left\|X_t^{(b)}\right\|^n\right] \;\le\; k\left(1 + Te^{nKT}\left(\|b\|_T^n + K^n\right)\right).$$

Proof. Let $b^1(t,x) := b(t,x)$ and $b^2(t,x) = Kx$, and denote by Y, Z the diffusions associated with b^1, b^2. By Lemma 8.6 we have for $t \in [0,T]$

$$\begin{aligned}
&\mathbb{E}\left[\|Y_t\|^n\right] \\
&\le 2^n\left(\mathbb{E}\left[\|Z_t\|^n\right] + \mathbb{E}\left[\|Y_t - Z_t\|^n\right]\right) \\
&\le 2^n\,\mathbb{E}\left[\|Z_t\|^n\right] + 2^n e^{nKT}\left\|b^1 - b^2\right\|_T^n\,\mathbb{E}\left[\int_0^t (1+\|Z_s\|^{2q})^n ds\right] \\
&\le 2^n\left(1 + \mathbb{E}\left[\|Z_t\|^{2qn}\right]\right) + 2^n e^{nKT} t\left(\left\|b^1\right\|_T + \left\|b^2\right\|_T\right)^n \sup_{0\le s\le T}\mathbb{E}\left[(1+\|Z_s\|^{2q})^n\right] \\
&\le 8^n\left(1 + \sup_{0\le s\le T}\mathbb{E}\left[\|Z_s\|^{2qn}\right]\right)\left(1 + te^{nKT}\left(\left\|b^1\right\|_T^n + \left\|b^2\right\|_T^n\right)\right).
\end{aligned}$$

Due to the assumption $\mathbb{E}\left[\left\|X_0^{(b)}\right\|^{2qn}\right] < \infty$, the constant

$$k(n,T) = 8^n\left(1 + \sup_{0\le s\le T}\mathbb{E}\left[\|Z_s\|^{2qn}\right]\right)$$

is finite by Lemma 8.7. Furthermore, we have $\|b^2\|_T \le K$, i.e. the lemma is proved. □

Now we are in a position to establish the local Lipschitz continuity of the operator Γ. The explicit expression for the Lipschitz constant shows that Γ will be a contraction on a sufficiently small time interval.

8.9 Lemma. *Let $b^1, b^2 \in \Lambda_T$, and denote by Y,Z the corresponding diffusions as in Lemma 8.6. For $i \in \mathbb{N}$ let*

$$m_i(T) = \sup_{0\le t\le T} \mathbb{E}\left[\, \|Y_t\|^i \,\right] \qquad \text{and} \qquad n_i(T) = \sup_{0\le t\le T} \mathbb{E}\left[\, \|Z_t\|^i \,\right].$$

There exists a constant $k = k(m_{4q}(T), n_{4q}(T))$ such that

$$\|\Gamma b_1 - \Gamma b_2\|_T \le k\sqrt{T} e^{KT} \|b^1 - b^2\|_T.$$

Proof. From Lemma 8.4 c) and the Cauchy-Schwarz inequality follows that

$$\begin{aligned}
\left\|\Gamma b^1(t,x) - \Gamma b^2(t,x)\right\| &\le \mathbb{E}\left[\, \|\Phi(x - Y_t) - \Phi(x - Z_t)\| \,\right] \\
&\le K_1\Big(1 + \|x\|^r\Big)\, \mathbb{E}\left[\, \|Y_t - Z_t\| \left(1 + \|Y_t\|^r + \|Z_t\|^r\right)\right] \\
&\le K_1\Big(1 + \|x\|^r\Big)\sqrt{\mathbb{E}\left[\, \|Y_t - Z_t\|^2 \,\right] \mathbb{E}\left[\left(1 + \|Y_t\|^r + \|Z_t\|^r\right)^2\right]},
\end{aligned}$$

where $K_1 = \max(K, 2^{r+1})$. By Lemma 8.6, since $(1 + x)^2 \le 2(1 + x^2)$, we have

$$\begin{aligned}
\mathbb{E}\left[\, \|Y_t - Z_t\|^2 \,\right] &\le e^{2KT} \left\|b^1 - b^2\right\|_T^2 \, \mathbb{E}\left[\left(\int_0^T \left(1 + \|Z_s\|^{2q}\right) ds\right)^2\right] \\
&\le e^{2KT} \left\|b^1 - b^2\right\|_T^2 \int_0^T \mathbb{E}\left[(1 + \|Z_s\|^{2q})^2\right] ds \\
&\le 2T\, e^{2KT} \left\|b^1 - b^2\right\|_T^2 \left(1 + \sup_{0\le s\le T} \mathbb{E}\left[\, \|Z_s\|^{4q} \,\right]\right).
\end{aligned}$$

Moreover, using the inequality $(a + b)^2 \le 2(a^2 + b^2)$, we deduce that

$$\begin{aligned}
\mathbb{E}\left[\left(1 + \|Y_t\|^r + \|Z_t\|^r\right)^2\right] &\le 2\Big(1 + 2\,\mathbb{E}\left[\, \|Y_t\|^{2r} + \|Z_t\|^{2r} \,\right]\Big) \\
&\le 10\Big(1 + \mathbb{E}\left[\, \|Y_t\|^{4q} + \|Z_t\|^{4q} \,\right]\Big),
\end{aligned}$$

where we exploited that $2q > r$ implies $\mathbb{E}\left[\, \|Y_t\|^{2r}\right] \le 1 + \mathbb{E}\left[\, \|Y_t\|^{4q}\right]$, and likewise for the moment of Z_t. By combining all these estimates, we find that

$$\begin{aligned}
\frac{\|\Gamma b^1(t,x) - \Gamma b^2(t,x)\|}{1 + \|x\|^{2q}} &\le 2K_1\sqrt{5T}\, e^{KT} \left\|b^1 - b^2\right\|_T \frac{1 + \|x\|^r}{1 + \|x\|^{2q}} \\
&\quad \times \left(1 + \sup_{0\le s\le T} \mathbb{E}\left[\, \|Z_s\|^{4q} \,\right]\right)^{1/2} \left(1 + \mathbb{E}\left[\, \|Y_t\|^{4q} + \|Z_t\|^{4q} \,\right]\right)^{1/2}.
\end{aligned}$$

Hence, if we set $k := 4K_1\sqrt{5}\Big\{\Big(1+n_{4q}(T)\Big)\Big(1+m_{4q}(T)+n_{4q}(T)\Big)\Big\}^{1/2}$, we may conclude that

$$\left\|\Gamma b^1 - \Gamma b^2\right\|_T \;\le\; k\sqrt{T}\, e^{KT} \left\|b^1 - b^2\right\|_T ,$$

i.e. k is the desired constant. □

The next proposition shows that the restriction of Γ to a suitable subset of the function space Λ_T is a contractive mapping, which allows us to construct a solution on a small time interval.

8.10 Proposition. *For $\nu > 0$ let $\Lambda_T^\nu = \{b \in \Lambda_T : \|b\|_T \le \nu\}$. Assume that the initial condition X_0 satisfies $\mathbb{E}\left[\|X_0\|^{2qn}\right] < \infty$ for some $n \ge 4q$. There exists $\nu_0 > 0$ such that for any $\nu \ge \nu_0$ there exists $T = T(\nu) > 0$ such that the following holds true:*

a) $\Gamma(\Lambda_T^\nu) \subset \Lambda_T^\nu$, and the Lipschitz constant of $\Gamma|\Lambda_T^\nu$ is less than $\frac{1}{2}$.

b) There exists a strong solution to (8.10), (8.11) on $[0, T]$ which satisfies

$$\sup_{0\le t\le T} \mathbb{E}\left[\left\|X_t^{(b)}\right\|^n\right] \;\le\; k\Big(1 + Te^{nKT}\Big(\nu^n + K^n\Big)\Big),$$

where $k = k(n, T)$ is the constant introduced in Lemma 8.8.

Proof. Let $b \in \Lambda_T$, and let $X = X^{(b)}$ and $m_i(T) = \sup_{0\le t\le T} \mathbb{E}\left[\|X_t\|^i\right]$ for $i \in \mathbb{N}$. By Lemma 8.8 the condition $\mathbb{E}\left[\|X_0\|^{2qn}\right] < \infty$ implies $m_i(T) < \infty$ for $T > 0$ and $i \le n$. Moreover, Lemma 8.4 shows that

$$\begin{aligned}\|\Gamma b(t,x)\| \;&\le\; 2K + (K + 2^{r+1})\Big(\|x\|^{r+1} + \mathbb{E}\left[\|X\|^{r+1}\right]\Big)\\ &\le\; \tilde K(1 + \|x\|^{r+1})\Big(1 + \mathbb{E}\left[\|X_t\|^{r+1}\right]\Big),\end{aligned}$$

where $\tilde K = 2K + 2^{r+1}$. Consequently, by definition of $\|\cdot\|_T$,

$$\|\Gamma b\|_T \le 2\tilde K(1 + m_{r+1}(T)), \quad t \le T. \tag{8.30}$$

By Lemma 8.8 there exists $k = k(r+1, T) > 0$ such that

$$m_{r+1}(T) \;\le\; k\Big(1 + Te^{(r+1)KT}\Big(\|b\|_T^{r+1} + K^{r+1}\Big)\Big). \tag{8.31}$$

This inequality, together with (8.30), is the key for finding a suitable subset of Λ_T on which Γ is contractive. The r.h.s. of (8.31) converges to k as $T \to 0$, and this convergence is uniform w.r.t. $b \in \Lambda_T^\nu$ for each $\nu > 0$. The dependence of the limiting constant k on T imposes no problem here; just fix $k = k(r+1, T_0) > 0$ for some T_0 and use the fact that (8.31) is valid for all $T \le T_0$, as the proof of Lemma 8.8 shows. Thus, we may fix $\nu_0 > 2\tilde K(1 + k)$ and deduce that for any $\nu > \nu_0$ we can find

$T_0 = T_0(\nu)$ such that $\|b\|_T \leq \nu$ implies $\|\Gamma b\|_T \leq \nu$ for $T \leq T_0$. Moreover, by Lemma 8.5, Γb satisfies all the conditions as required for it to belong to Λ_T, i.e. Γ maps Λ_T^ν into itself for all $T \leq T_0$. Additionally, the assumption $n \geq 4q$ implies that $m_{4q}(T)$ is uniformly bounded for all b in Λ_T^ν, and Lemma 8.9 shows that, by eventually decreasing T_0, we can achieve that Γ is a contraction on Λ_T^ν with Lipschitz constant less than $\frac{1}{2}$, i.e. a) is established.

In order to prove b), the existence of a strong solution on the time interval $[0, T]$ for $T \leq T_0$, we iterate the drift through Γ. Let $b_0 \in \Lambda_T^\nu$, and define

$$b_{n+1} := \Gamma b_n \qquad \text{for } n \in \mathbb{N}_0 .$$

The contraction property of Γ yields $\|b_{n+1} - b_n\|_T \leq 2^{-n} \|b_1 - b_0\|_T$ for all n, and therefore

$$\sum_{n=0}^{\infty} \|b_{n+1} - b_n\|_T < \infty,$$

which entails that (b_n) is a Cauchy sequence w.r.t. $\|\cdot\|_T$. By definition of $\|\cdot\|_T$, (b_n) converges pointwise to a continuous function $b = b(t, x)$ with $\|b\|_T < \infty$. It remains to verify that the limit is again an element of Λ_T. In order to see that it is locally Lipschitz, let $X^{(n)} := X^{(b_n)}$. As in the proof of Lemma 8.5, we have for $x, y \in B_R(0)$

$$\begin{aligned} \|\Gamma b_n(t,x) - \Gamma b_n(t,y)\| &\leq \mathbb{E}\left[\left\|\Phi(x - X_t^{(n)}) - \Phi(y - X_t^{(n)})\right\|\right] \\ &\leq \|x - y\| \left[K + 2^{r+1}\Big(R^r + \mathbb{E}\left[\left\|X_t^{(n)}\right\|^r\right]\Big)\right]. \end{aligned}$$

Since $\|b_n\|_T \leq \nu$ for all n, (8.31) yields

$$\sup_{n \in \mathbb{N}} \sup_{0 \leq t \leq T} \mathbb{E}\left[\left\|X_t^{(n)}\right\|^r\right] \leq k\Big(1 + Te^{(r+1)KT}\Big(\nu^{r+1} + K^{r+1}\Big)\Big).$$

Therefore, we may send $n \to \infty$ to conclude that b is locally Lipschitz. b being the pointwise limit of the b_n, it inherits the polynomial growth property and the dissipativity condition as stated in Lemma 8.5 b) and c). (Notice that we may not invoke Lemma 8.5 at this stage.)

It remains to show that the diffusion $X = X^{(b)}$ associated to b has the desired properties. Note first that the existence of X is guaranteed by the classical result of Corollary 2.2. Since $\Gamma b = b$, which means that

$$b(t,x) = \Gamma b(t,x) = \mathbb{E}\left[\Phi(x - X_t^{(b)})\right]$$

for $t \in [0, T]$ and $x \in \mathbb{R}^d$, X is the diffusion with interaction drift b. The boundedness of its moments is again a consequence of Lemma 8.8. $\square$

Let us recall the essentials of the construction carried out so far. We have shown the existence of a solution to (8.9) on a small time interval $[0, T]$. For the moments of order n to be finite, one needs integrability of order $2qn$ for the initial condition. Moreover, the parameter n needs to be larger or equal to $4q$ in order for the fixed point argument of Proposition 8.10 to work. Observe that the condition $n \geq 4q$ appears first in this Proposition, since this is the first time the process is coupled to its own drift, while in all previous statements the finiteness of moments is guaranteed by the comparison against the diffusion Z, which is governed by a linear drift term.

In order to find a solution that exists for all times, we need to carefully extend the constructed pair (X, b) beyond the time horizon T. Although non-explosion and finiteness of moments would be guaranteed for all T by Corollary 2.2 and Lemma 8.8, we have to take care of the fact that the drift itself is defined only on the time interval $[0, T]$. With sufficiently strong integrability assumptions for X_0 one could perform the same construction on the time intervals $[T, 2T]$, $[2T, 3T]$ and so on, but one loses an integrability order $2q$ in each time step of length T.

For that reason we need better control of the moments of X over the whole time axis, which is achieved by the following a posteriori estimate.

8.11 Proposition. *Let $m \in \mathbb{N}$, $m \geq 4q^2$, such that $\mathbb{E}\left[\, \|X_0\|^{2m} \,\right] < \infty$. For each $n \in \{1, \ldots, m\}$ there exists a constant $\alpha = \alpha(n) > 0$ such that the following holds true for all $T > 0$: if X solves (8.9) on $[0, T]$, then*

$$\sup_{0 \leq t \leq T} \mathbb{E}\left[\, \|X_t\|^{2n} \,\right] \leq \alpha(n).$$

Proof. Let $f_n(t) = \mathbb{E}[\|X_t\|^{2n}]$, and let $b(t, x) = \mathbb{E}\left[\Phi(x - X_t)\right]$. We proceed in several steps.

Step 1: Boundedness in L^2. By Lemma 8.8 we know that $\sup_{0 \leq t \leq T} f_1(t) < \infty$. The only point is to show that the bound may be chosen independent of T. By Itô's formula we have

$$f_1(t) = \mathbb{E}\left[\, \|X_0\|^2 \,\right] + \varepsilon t d + 2 \int_0^t \mathbb{E}\left[\langle X_s, V(X_s)\rangle\right] ds - 2 \int_0^t \mathbb{E}\left[\langle X_s, b(s, X_s)\rangle\right] ds.$$

Let us first estimate the last term that contains the interaction drift b. By its definition, we may take an independent copy $\tilde{X}$ of X, to write

$$\begin{aligned}
2\,\mathbb{E}\left[\langle X_s, b(s, X_s)\rangle\right] &= 2\,\mathbb{E}\left[\langle X_s, \Phi(X_s - \tilde{X}_s)\rangle\right] \\
&= \mathbb{E}\left[\langle X_s, \Phi(X_s - \tilde{X}_s)\rangle\right] - \mathbb{E}\left[\langle \tilde{X}_s, \Phi(X_s - \tilde{X}_s)\rangle\right] \\
&= \mathbb{E}\left[\langle X_s - \tilde{X}_s, \Phi(X_s - \tilde{X}_s)\rangle\right] \;\geq 0
\end{aligned}$$

where the last inequality is due to (8.13). In order to estimate the other integral, let $R \geq R_1$. Using (8.18) and the local Lipschitz property of V, we see that

$$\begin{aligned}
\mathbb{E}\left[\langle X_s, V(X_s)\rangle\right] \leq & -\eta\,\mathbb{E}\left[\,\|X_s\|^2\,\mathbf{1}_{\{\|X_s\|>R\}}\right] \\
& + \mathbb{E}\left[(K\,\|X_s\|^2 + \|V(0)\|\,\|X_s\|)\mathbf{1}_{\{\|X_s\|\leq R\}}\right] \\
\leq & -\eta\,\mathbb{E}\left[\,\|X_s\|^2\,\right] + (\eta+K)R^2 + \|V(0)\|\,R \\
= & -\eta f_1(s) + R(\|V(0)\| + R(\eta+K)).
\end{aligned}$$

Obviously, f_1 is differentiable, and summing up these bounds yields

$$f_1'(t) \leq \varepsilon d - 2\eta f_1(t) + 2R(\|V(0)\| + R(\eta+K)).$$

Thus, there exists $\gamma > 0$ such that $\{t \in [0,T] :\ f_1(t) \geq \gamma\} \subset \{t \in [0,T] :\ f_1'(t) \leq 0\}$, which implies $f_1(t) \leq f_1(0) \vee \gamma$ for all $t \in [0,T]$. This is the claimed bound, since γ is independent of T.

Step 2: Moment bound for the convolution. Let $\tilde{X}$ be an independent copy of X, i.e. a solution of (8.9) driven by a Brownian motion that is independent of W. In this step we shall prove that $\mathbb{E}[\|X_t - \tilde{X}_t\|^{2n}]$ is uniformly bounded w.r.t. time.
Let $R \geq R_1$, and let $\tau = \inf\{t \geq 0 :\ \|X_t - \tilde{X}_t\| \geq R\}$, $g_n(t) = \mathbb{E}[\|X_t - \tilde{X}_t\|^{2n}\mathbf{1}_{\{t<\tau\}}]$ and $w_n(t) = \mathbb{E}[\|X_{t\wedge\tau} - \tilde{X}_{\tau\wedge t}\|^{2n}]$. Then $w_n(t) = g_n(t) + R^{2n}\,\mathbb{P}(t \geq \tau)$. Furthermore, using the SDE (8.9) for both X and $\tilde{X}$, applying Itô's formula to the difference and taking expectations, we obtain for $n \geq 1$

$$\begin{aligned}
w_n(t) \;=\; & \mathbb{E}[\|X_0 - \tilde{X}_0\|^{2n}] + \varepsilon(dn+n-1)\;\mathbb{E}\left[\int_0^{t\wedge\tau} \|X_s - \tilde{X}_s\|^{2n-2}\,ds\right] \\
& + 2n\;\mathbb{E}\left[\int_0^{t\wedge\tau} \|X_s - \tilde{X}_s\|^{2n-2}\langle X_s - \tilde{X}_s, V(X_s) - V(\tilde{X}_s)\rangle\,ds\right] \\
& - 2n\;\mathbb{E}\left[\int_0^{t\wedge\tau} \|X_s - \tilde{X}_s\|^{2n-2}\langle X_s - \tilde{X}_s, b(s,X_s) - b(s,\tilde{X}_s)\rangle\,ds\right].
\end{aligned}$$

The last term is negative by Lemma 8.5, which yields together with (8.16), (8.17) and Hölder's inequality

$$\begin{aligned}
w_n'(t) \;\leq\; & \varepsilon(dn+n-1)\;\mathbb{E}\left[\|X_t - \tilde{X}_t\|^{2n-2}\mathbf{1}_{\{t<\tau\}}\right] \\
& + 2n\;\mathbb{E}\left[\|X_t - \tilde{X}_t\|^{2n-2}\langle X_t - \tilde{X}_t, V(X_t) - V(\tilde{X}_t)\rangle\mathbf{1}_{\{t<\tau\}}\right] \\
\leq\; & \varepsilon(dn+n-1)\,g_{n-1}(t) + 2n(K+\eta)\;\mathbb{E}\left[\|X_t - \tilde{X}_t\|^{2n}\mathbf{1}_{\left\{\|X_t - \tilde{X}_t\|\leq R_1\,;\,\tau>t\right\}}\right] \\
& - 2n\eta\;\mathbb{E}\left[\|X_t - \tilde{X}_t\|^{2n}\mathbf{1}_{\{t<\tau\}}\right] \\
\leq\; & \varepsilon(dn+n-1)\,g_n(t)^{1-\frac{1}{n}} + 2n(K+\eta)R_1^{2n} - 2n\eta g_n(t).
\end{aligned}$$

As in the first step, there exists some constant $\delta > 0$ such that $\{t \in [0,T] : g_n(t) > \delta\} \subset \{t \in [0,T] : w_n'(t) < 0\}$. Since $w_n - g_n$ is non-decreasing this implies $g_n(t) \leq g_n(0) \vee \delta$ for all $t \in [0,T]$. Moreover, δ depends only on the constants appearing in the last inequality and is independent of the localization parameter. Hence, by monotone convergence, we have

$$\mathbb{E}[\|X_t - \tilde{X}_t\|^{2n}] \ \leq \ \mathbb{E}[\|X_0 - \tilde{X}_0\|^{2n}] \vee \delta, \qquad t \in [0,T].$$

Step 3: Bound for the centered moments of X. In this step we shall prove that the moments of $Y_t := X_t - \mathbb{E}[X_t]$ are uniformly bounded. We proceed by induction. The second moments of X are uniformly bounded by the first step, so are those of Y. Assume the moments of order $2n$ are uniformly bounded by $\gamma_n > 0$. If $n+1 \leq m$, we may invoke step 2, to find $\delta_{n+1} > 0$ such that $\mathbb{E}[\|X_t - \tilde{X}_t\|^{2n+2}] \leq \delta_{n+1}$ for $t \in [0,T]$. Now we make the following observation. If ξ, $\tilde{\xi}$ are independent, real-valued copies of each other with $\mathbb{E}[\xi] = 0$, then

$$\mathbb{E}\left[(\xi - \tilde{\xi})^{2n+2}\right] \ = \ 2\,\mathbb{E}\left[\xi^{2n+2}\right] + \sum_{k=2}^{2n} \binom{2n+2}{k} (-1)^k\, \mathbb{E}\left[\xi^k\right] \mathbb{E}\left[\xi^{2n+2-k}\right],$$

and therefore

$$\begin{aligned} 2\,\mathbb{E}\left[\xi^{2n+2}\right] \ &\leq \ \mathbb{E}\left[(\xi - \tilde{\xi})^{2n+2}\right] + \sum_{k=2}^{2n} \binom{2n+2}{k} \left|\mathbb{E}\left[\xi^k\right] \mathbb{E}\left[\xi^{2n+2-k}\right]\right| \\ &\leq \ \mathbb{E}\left[(\xi - \tilde{\xi})^{2n+2}\right] + 2^{2n+2}\Big(1 + \mathbb{E}\left[\xi^{2n}\right]\Big)^2. \end{aligned}$$

Let us apply this to the components of Y, and denote them by $Y^1, \ldots, Y^d$. We obtain for $t \in [0,T]$

$$\begin{aligned} 2\,\mathbb{E}\Big[\,\|Y_t\|^{2n+2}\,\Big] \ &\leq \ 2d^{n+1}\,\mathbb{E}\Big[\sum_{j=1}^{d} (Y_t^j)^{2n+2}\Big] \\ &\leq \ d^{n+1} \sum_{j=1}^{d} \mathbb{E}\left[(X_t^j - \tilde{X}_t^j)^{2n+2}\right] + 2^{2n+2}\Big(1 + \mathbb{E}\left[(Y_t^j)^{2n}\right]\Big)^2 \\ &\leq \ d^{n+2}\bigg(\,\mathbb{E}\left[\|X_t - \tilde{X}_t\|^{2n+2}\right] + 2^{2n+2}\Big(1 + \mathbb{E}\Big[\,\|Y_t\|^{2n}\,\Big]\Big)^2\bigg) \\ &\leq \ d^{n+2}\bigg(\delta_{n+1} + 2^{2n+2}\Big(1 + \gamma_n\Big)^2\bigg), \end{aligned}$$

which is a uniform bound for the order $2(n+1)$.

Step 4: Bound for the moments of X. In the fourth and final step, we prove the announced uniform bound for the moments of X. It follows immediately from the inequality

$$\mathbb{E}\Big[\,\|X_t\|^{2n}\,\Big] \ \leq \ 2^{2n}\Big(\,\mathbb{E}\Big[\,\|X_t - \mathbb{E}[X_t]\|^{2n}\,\Big] + \|\mathbb{E}[X_t]\|^{2n}\,\Big).$$

The last term satisfies $\|\mathbb{E}[X_t]\|^{2n} \leq f_1(t)^n$, which is uniformly bounded according to step 1, and the centered moments of order $2n$ are uniformly bounded by step 3 whenever $n \leq m$. $\square$

The results concerning the existence of X^ε are summarized in the following theorem.

8.12 Theorem. *Let $q := \left[\frac{r}{2} + 1\right]$, and let X_0 be a random initial condition such that $\mathbb{E}\left[\, \|X_0\|^{8q^2} \,\right] < \infty$. Then there exists a drift term $b(t,x) = b^{\varepsilon, X_0}(t,x)$ such that* (8.10) *admits a unique strong solution X^ε that satisfies* (8.11)*, and X^ε is the unique strong solution of* (8.9)*. Moreover, we have for all $n \in \mathbb{N}$*

$$\sup_{t \geq 0} \mathbb{E}\left[\, \|X_t^\varepsilon\|^{2n} \,\right] < \infty \tag{8.32}$$

whenever $\mathbb{E}\left[\, \|X_0^\varepsilon\|^{2n} \,\right] < \infty$. In particular, if X_0 is deterministic, then X^ε is bounded in $L^p(\mathbb{P} \otimes \boldsymbol{\lambda}_{[0,T]})$ for all $p \geq 1$.

Proof. In a first step, we prove uniqueness on a small time interval. Let $\tilde{K} = 2K + 2^{r+1}$, and choose $\alpha(q) > 0$ according to Proposition 8.11. By Proposition 8.10 there exist $\nu \geq 2\tilde{K}(2 + \alpha(q))$, $T = T(\nu) > 0$ and $b \in \Lambda_T^\nu$ such that $\Gamma b = b$, i.e. $X = X^{(b)}$ is a strong solution of (8.9) on $[0,T]$. Assume Y is another solution of (8.9) on $[0,T]$ starting at X_0 such that $m_{2q}(T) := \sup_{0 \leq t \leq T} \mathbb{E}[\|Y_t\|^{2q}] < \infty$, and let $c(t,x) = \mathbb{E}\left[\Phi(x - Y_t)\right]$. Then $c \in \Lambda_T$ by Lemma 8.5, and $\Gamma c = c$. Moreover, it follows from (8.30) and Proposition 8.11 that

$$\|c\|_T \;\leq\; 2\tilde{K}(2 + m_{2q}(T)) \;\leq\; 2\tilde{K}(2 + \alpha(q)) \;\leq\; \nu,$$

i.e. $c \in \Lambda_T^\nu$. Hence c is the unique fixed point of $\Gamma | \Lambda_T^\nu$. Thus $c = b$, and Corollary 2.2 yields $X = Y$.

In the second step, we show the existence of a unique solution on $[0,\infty)$. Let

$$U := \sup\Big\{T > 0 : \text{(8.9) admits a unique strong solution } X \text{ on } [0,T], \\ \sup_{0 \leq t \leq T} \mathbb{E}\left[\, \|X_t\|^{2q} \,\right] < \infty\Big\}.$$

By the first step we know that $U > 0$. Assume $U < \infty$. As in the first step, choose $\alpha(4q^2) > 0$ according to Proposition 8.11, and then fix $\tilde{\nu} \geq 2\tilde{K}(2 + \alpha(4q^2))$ and $\tilde{T} = \tilde{T}(\tilde{\nu}) > 0$ that satisfy Proposition 8.10. Let $0 < \delta < \min(U, \tilde{T}/2)$, and fix $T \in]U - \delta, U[$. There exists a unique strong solution X on $[0,T]$, and $\mathbb{E}[\|X_T\|^{8q^2}] < \infty$ by Proposition 8.11. Now consider equation (8.9) on $[T,\infty)$ with initial datum X_T. As in the first step, we may find a unique strong solution on $[T, T + \tilde{T}]$. But this is a contradiction since $T + \tilde{T} > U$. Consequently, $U = \infty$, and (8.32) holds by Proposition 8.11. $\square$

Chapter 9

Large deviations for self-stabilizing diffusions

Let us now turn to the large deviations behavior of the self-stabilizing diffusion X^ε given by the SDE (8.4), i.e.

$$dX_t^\varepsilon = V(X_t^\varepsilon)\,dt - \int_{\mathbb{R}^d} \Phi(X_t^\varepsilon - x)\,du_t^\varepsilon(x)\,dt + \sqrt{\varepsilon}dW_t, \quad t \geq 0,\ X_0 = x_0 \in \mathbb{R}^d. \quad (9.1)$$

The heuristics underlying large deviations theory is to identify a deterministic path around which the diffusion is concentrated with overwhelming probability, so that the stochastic motion can be seen as a small random perturbation of this deterministic path. This means in particular that the law u_t^ε of X_t^ε is close to some Dirac mass if ε is small.

In this chapter, we therefore proceed in two steps towards the aim of proving a large deviations principle for X^ε. In a first step we 'guess' the deterministic limit around which X^ε is concentrated for small ε, and replace u_t^ε by its suspected limit, i.e. we approximate the law of X^ε (Section 9.1). This way we circumvent the difficulty of the dependence on the law of X^ε – the self-interaction term – and obtain a diffusion which is defined by means of a classical SDE. We then prove in the second step that this diffusion is exponentially equivalent to X^ε, i.e. it has the same large deviations behavior. This is done in Section 9.2 and involves pathwise comparisons. Finally, in Section 9.3, we prove an exponential approximation for X^ε in case the underlying deterministic geometry admits a stable fixed point.

9.1 Small noise asymptotics of the interaction drift

The limiting behavior of the diffusion X^ε can be guessed in the following way. As pointed out before, the laws u_t^ε should tend to a Dirac measure in the small noise limit, and since $\Phi(0) = 0$ the interaction term will vanish in the limiting equation. Therefore, the diffusion X^ε is a small random perturbation of the deterministic motion ψ, given as the solution of the deterministic equation

$$\dot{\psi}_t = V(\psi_t), \quad \psi_0 = x_0, \tag{9.2}$$

and the large deviations principle will describe the asymptotic deviation of X^ε from this path. Much like in the case of gradient type systems, the dissipativity condition (8.18) guarantees non-explosion of ψ. Indeed, since $\frac{d}{dt}\|\psi_t\|^2 = 2\langle\psi_t, \dot{\psi}_t\rangle = 2\langle\psi_t, V(\psi_t)\rangle$, the derivative of $\|\psi_t\|^2$ is negative for large values of $\|\psi_t\|$ by (8.18), so ψ is bounded. In the sequel we shall write $\psi_t(x_0)$ if we want to stress the dependence on the initial condition.

We have to control the diffusion's deviation from this deterministic limit on a finite time interval. An a priori estimate is provided by the following lemma, which gives an L^2-bound for this deviation. For notational convenience, we suppress the ε-dependence of the diffusion in the sequel, but keep in mind that all processes depend on ε.

9.1 Lemma. *Let $Z_t := X_t - \psi_t(x_0)$. Then*

$$\mathbb{E}\,\|Z_t\|^2 \le \varepsilon t d\, e^{2Kt},$$

where K is the constant introduced in Lemma 8.3. In particular, we have $Z \to 0$ as $\varepsilon \to 0$ in $L^p(\mathbb{P} \otimes \boldsymbol{\lambda}_{[0,T]})$ for all $p \ge 1$ and $T > 0$. This convergence is locally uniform w.r.t. the initial condition x_0.

Proof. By Itô's formula we have

$$\begin{aligned}\|Z_t\|^2 &= 2\sqrt{\varepsilon}\int_0^t \langle Z_s, dW_s\rangle - 2\int_0^t \Big\langle Z_s, b^{\varepsilon,x_0}(s, Z_s + \psi_s(x_0))\Big\rangle\, ds \\ &\quad + 2\int_0^t \Big\langle Z_s, V(Z_s + \psi_s(x_0)) - V(\psi_s(x_0))\Big\rangle\, ds + \varepsilon t d.\end{aligned}$$

Since X and thus Z is square-integrable by Theorem 8.12, the stochastic integral in this equation is a martingale. Now consider the second term containing the interaction drift b^{ε,x_0}. Let $\nu_s = \mathbb{P} \circ Z_s^{-1}$ denote the law of Z_s, and recall Assumption 8.2 ii) about the interaction function Φ. The latter implies

$$\begin{aligned}2\,\mathbb{E}\Big\langle Z_s, b^{\varepsilon,x_0}(s, Z_s + \psi_s(x_0))\Big\rangle &= 2\int \Big\langle z, \mathbb{E}\Big[\Phi(z + \psi_s(x_0) - X_s)\Big]\Big\rangle\, \nu_s(dz) \\ &= 2\int\int \Big\langle z, \Phi(z-y)\Big\rangle\, \nu_s(dy)\, \nu_s(dz) \\ &= \int\int \Big\langle z-y, \Phi(z-y)\Big\rangle\, \nu_s(dy)\, \nu_s(dz) \;\ge\; 0.\end{aligned}$$

Hence by the growth condition (8.16) for V

$$\begin{aligned}\mathbb{E}\|Z_t\|^2 &\le 2\int_0^t \mathbb{E}\Big\langle Z_s, V(Z_s+\psi_s(x_0)) - V(\psi_s(x_0))\Big\rangle\,ds + \varepsilon t d \\ &\le 2K\int_0^t \mathbb{E}\|Z_s\|^2\,ds + \varepsilon t d,\end{aligned}$$

and Gronwall's lemma yields

$$\mathbb{E}\|Z_t\|^2 \le \varepsilon t d\, e^{2Kt}.$$

This is the claimed bound. For the L^p-convergence observe that this bound is independent of the initial condition x_0. Moreover, the argument of Proposition 8.11 shows that $\sup\Big\{\mathbb{E}\Big(\|X_t\|^p\Big) : 0\le t\le T,\ x_0\in L,\ 0<\varepsilon<\varepsilon_0\Big\} < \infty$ holds for compact sets L and $\varepsilon_0>0$. This implies that Z is bounded in $L^p(\mathbb{P}\otimes\boldsymbol{\lambda}_{[0,T]})$ as $\varepsilon\to 0$, uniformly w.r.t. $x_0\in L$. Now the L^p-convergence follows from the Vitali convergence theorem. □

9.2 Corollary. *For any $T>0$ we have*

$$\lim_{\varepsilon\to 0} b^{\varepsilon,x_0}(t,x) = \Phi(x-\psi_t(x_0)),$$

uniformly w.r.t. $t\in[0,T]$ and w.r.t. x and x_0 on compact subsets of $\mathbb{R}^d$.

Proof. The growth condition on Φ and the Cauchy-Schwarz inequality yield

$$\begin{aligned}&\Big\|b^{\varepsilon}(t,x) - \Phi(x-\psi_t(x_0))\Big\|^2 \\ &\quad\le \mathbb{E}\Big[\|X_t-\psi_t(x_0)\|\Big(K+\|x-X_t\|^r+\|x-\psi_t(x_0)\|^r\Big)\Big]^2 \\ &\quad\le \mathbb{E}\Big[\|X_t-\psi_t(x_0)\|^2\Big]\,\mathbb{E}\Big[\Big(K+\|x-X_t\|^r+\|x-\psi_t(x_0)\|^r\Big)^2\Big].\end{aligned}$$

The first expectation on the r.h.s. of this inequality tends to zero by Lemma 9.1. Since X is bounded in $L^{2r}(\mathbb{P})$, uniformly w.r.t. x_0 on compact sets, the claimed convergence follows. □

In a next step we replace the diffusion's law in (9.1) by its limit, the Dirac measure in $\psi_t(x_0)$. Before doing so, let us introduce a slight generalization of X.
Theorem 8.12 implies that X is a time inhomogeneous Markov process. The diffusion X, starting at time $s\ge 0$, is given as the unique solution of the stochastic integral equation

$$X_t = X_s + \int_s^t [V(X_u) - b^{\varepsilon,x_0}(u,X_u)]\,du + \sqrt{\varepsilon}(W_t-W_s),\quad t\ge s.$$

By shifting the starting time back to the origin, this equation translates into

$$X_{t+s} = X_s + \int_0^t [V(X_{u+s}) - b^{\varepsilon,x_0}(u+s, X_{u+s})]\,du + \sqrt{\varepsilon}W_t^s, \quad t \geq 0,$$

where W^s is the Brownian motion given by $W_t^s = W_{t+s} - W_s$, which is independent of X_s. Since we are mainly interested in the law of X, we may replace W^s by W.

For an initial condition $\xi_0 \in \mathbb{R}^d$ and $s \geq 0$, we denote by ξ^{s,ξ_0} the unique solution of the equation

$$\xi_t = \xi_0 + \int_0^t [V(\xi_u) - b^{\varepsilon,x_0}(u+s, \xi_u)]\,du + \sqrt{\varepsilon}W_t, \quad t \geq 0. \tag{9.3}$$

Note that $\xi^{0,x_0} = X$, and that ξ^{s,ξ_0} has the same law as X_{t+s}, given that $X_s = \xi_0$. The interpretation of b^{ε,x_0} as an interaction drift is lost in this equation, since b^{ε,x_0} does not depend on ξ^{s,ξ_0}.

Now recall that $b^{\varepsilon,x_0}(t,x) = \mathbb{E}\left\{\Phi(x - X_t^\varepsilon)\right\}$, which tends to $\Phi(x - \psi_t(x_0))$ by Corollary 9.2. This motivates the definition of the following analogue of ξ^{s,ξ_0}, in which u_t^ε is replaced by the Dirac measure in $\psi_t(x_0)$. We denote by $Y^{s,y}$ the solution of the equation

$$Y_t = y + \int_0^t V(Y_u) - \Phi(Y_u - \psi_{t+s}(x_0))\,du + \sqrt{\varepsilon}W_t, \quad t \geq 0. \tag{9.4}$$

This equation is an SDE in the classical sense, and it admits a unique strong solution by Corollary 2.2. Furthermore, it is known that $Y^{s,y}$ satisfies a large deviations principle in the space $C_{0T} = \{f : [0,T] \to \mathbb{R}^d \mid f \text{ is continuous}\}$, equipped with the topology of uniform convergence. This LDP describes the deviations of $Y^{s,y}$ from the deterministic system $\dot{\varphi}_t = V(\varphi_t) - \Phi(\varphi_t - \psi_{t+s}(x_0))$ with $\varphi_0 = y$. Observe that φ coincides with $\psi(x_0)$ in case $y = x_0$, and that non-explosion of φ is ensured by the dissipativity properties of V and Φ as follows. By (8.13) we have

$$\begin{aligned}
\frac{d}{dt}\|\varphi_t - \psi_{t+s}\|^2 &= 2\left\langle \varphi_t - \psi_{t+s}, \dot{\varphi}_t - \dot{\psi}_{t+s}\right\rangle \\
&= 2\left\langle \varphi_t - \psi_{t+s}, V(\varphi_t) - \Phi(\varphi_t - \psi_{t+s}) - V(\psi_{t+s})\right\rangle \\
&\leq 2\left\langle \varphi_t - \psi_{t+s}, V(\varphi_t) - V(\psi_{t+s})\right\rangle.
\end{aligned} \tag{9.5}$$

Since the last expression is negative for large values of $\|\varphi_t - \psi_{t+s}\|$ by (8.17), this means that $\varphi_t - \psi_{t+s}$ is bounded. But ψ is bounded, so φ is also bounded.

Recall that $\rho_{0T}(f,g) := \sup_{0\leq t\leq T}\|f - g\|$ $(f,g \in C_{0T})$ denotes the metric corresponding to uniform topology, and that H_{0T}^y is the Cameron-Martin space of absolutely continuous functions starting in y that possess square integrable derivatives.

9.3 Proposition. *The family* $(Y^{s,y})$ *satisfies a large deviations principle with good rate function*

$$I_{0T}^{s,y}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - \psi_{t+s}(x_0))\|^2 \, dt, & \text{if } \varphi \in H_{0T}^y, \\ \infty, & \text{otherwise }. \end{cases} \tag{9.6}$$

More precisely, for any closed set $F \subset C_{0T}$ *we have*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(Y^{s,y} \in F) \leq - \inf_{\varphi \in F} I_{0T}^{s,y}(\varphi),$$

and for any open set $G \subset C_{0T}$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(Y^{s,y} \in G) \geq - \inf_{\varphi \in G} I_{0T}^{s,y}(\varphi).$$

Proof. Let $a(t,y) := V(y) - \Phi(y - \psi_t)$, and denote by F the function that maps a path $g \in C_{0T}$ to the solution f of the ODE

$$f_t = x_0 + \int_0^t a(s, f_s) \, ds + g_t, \quad 0 \leq t \leq T.$$

Fix $g \in C_{0T}$, and let $R > 0$ such that the deterministic trajectory $\psi(x_0)$ as well as $f = F(g)$ stay in $B_R(0)$ up to time T. Note that non-explosion of f is guaranteed by dissipativity of a, much like in (9.5). Now observe that a is locally Lipschitz with constant $2K_{2R}$ on $B_R(0)$, uniformly w.r.t. $t \in [0,T]$. Thus, we have for $\tilde{g} \in C_{0T}$, $\tilde{f} = F(\tilde{g})$ such that $\tilde{f}$ does not leave $B_R(0)$ up to time T

$$\|f_t - \tilde{f}_t\| \leq 2K_{2R} \int_0^t \|f_s - \tilde{f}_s\| \, ds + \|g_t - \tilde{g}_t\| \,,$$

and Gronwall's lemma yields

$$\rho_{0T}(f, \tilde{f}) \ \leq \ \rho_{0T}(g, \tilde{g}) \ e^{2K_{2R}T},$$

i.e. F is continuous. Indeed, the last inequality shows that we do not have to presume that $\tilde{f}$ stays in $B_R(0)$, but that this is granted whenever $\rho_{0T}(g, \tilde{g})$ is sufficiently small.

Since F is continuous and $F(\sqrt{\varepsilon} W) = Y$, we may invoke Schilder's theorem and the contraction principle, to deduce that Y satisfies a large deviations principle with rate function

$$I_{0T}(\varphi) = \inf \left\{ \frac{1}{2} \int_0^T \|\dot{g}_t\|^2 \, dt : \ g \in H_{0T}^y, \ F(g) = \varphi \right\}.$$

This proves the LDP for $(Y^{s,y})$. $\square$

Notice that the rate function of Y measures distances from the deterministic solution ψ just as in the classical case without interaction, but the distance of φ from ψ is weighted by the interaction between the two paths.

As in the classical situation, one can associate to $Y^{s,y}$ two functions that determine the cost resp. energy of moving between points in the geometric landscape induced by the vector field V. For $t \geq 0$ the *cost function*

$$C^s(y,z,t) = \inf_{f \in C_{0t}:\ f_t = z} I_{0t}^{s,y}(f), \quad y, z \in \mathbb{R}^d,$$

determines the asymptotic cost for the diffusion $Y^{s,y}$ to move from y to z in time t, and the *quasi-potential*

$$Q^s(y,z) = \inf_{t>0} C^s(y,z,t)$$

describes its cost of going from y to z eventually.

9.2 The large deviations principle

We are now in a position to prove large deviations principles for ξ and X by showing that ξ and Y are close in the sense of large deviations.

9.4 Theorem. *For any $\varepsilon > 0$ let $x_0^\varepsilon, \xi_0^\varepsilon \in \mathbb{R}^d$ that converge to some $x_0 \in \mathbb{R}^d$ resp. $y \in \mathbb{R}^d$ as $\varepsilon \to 0$. Denote by X^ε the solution of* (9.1) *starting at x_0^ε. Let $s \geq 0$, and denote by ξ^ε the solution of* (9.3) *starting in ξ_0^ε with time parameter s, i.e.*

$$\xi_t^\varepsilon = \xi_0^\varepsilon + \int_0^t V(\xi_u^\varepsilon) - b^{\varepsilon,x_0}(u+s, \xi_u^\varepsilon)\, du + \sqrt{\varepsilon} W_t, \quad t \geq 0, \tag{9.7}$$

where $b^{\varepsilon,x_0}(t,x) = \mathbb{E}[\Phi(x - X_t^\varepsilon)]$.
Then the diffusions $(\xi^\varepsilon)_{\varepsilon>0}$ satisfy on any time interval $[0,T]$ a large deviations principle with good rate function (9.6).

Proof. We shall show that $\xi := \xi^\varepsilon$ is exponentially equivalent to $Y := Y^{s,y}$ as defined by (9.4), which has the desired rate function, i.e. we prove that for any $\delta > 0$ we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(\rho_{0T}(\xi, Y) \geq \delta) = -\infty. \tag{9.8}$$

Without loss of generality, we may choose $R > 0$ such that $x_0^\varepsilon, y \in B_R(0)$ and that $\psi_t(x_0)$ does not leave $B_R(0)$ up to time $s+T$, and denote by σ_R the first time at which ξ or Y exit from $B_R(0)$. Then for $t \leq \sigma_R$

$$\begin{aligned} \|\xi_t - Y_t\| \ &\leq \ \|\xi_0 - y\| + \int_0^t \|V(\xi_u) - V(Y_u)\|\, du \\ &\quad + \int_0^t \left\| b^{\varepsilon,x_0^\varepsilon}(u+s, \xi_u) - \Phi(Y_u - \psi_{u+s}(x_0)) \right\| du \end{aligned} \tag{9.9}$$

The first integral satisfies

$$\int_0^t \|V(\xi_u) - V(Y_u)\| \, du \leq K_R \int_0^t \|\xi_u - Y_u\| \, du, \quad t \leq \sigma_R,$$

due to the local Lipschitz assumption. Let us decompose the second integral. We have

$$\begin{aligned} \left\| b^{\varepsilon, x_0^\varepsilon}(u+s, \xi_u) - \Phi(Y_u - \psi_{u+s}(x_0)) \right\| \leq\ & \left\| b^{\varepsilon, x_0^\varepsilon}(u+s, \xi_u) - \Phi(\xi_u - \psi_{u+s}(x_0^\varepsilon)) \right\| \\ & + \|\Phi(\xi_u - \psi_{u+s}(x_0^\varepsilon)) - \Phi(\xi_u - \psi_{u+s}(x_0))\| \\ & + \|\Phi(\xi_u - \psi_{u+s}(x_0)) - \Phi(Y_u - \psi_{u+s}(x_0))\| . \end{aligned}$$

Bounds for the second and third term in this decomposition are easily derived. The last one is seen to be bounded by $K_{2R} \|\xi_u - Y_u\|$, since ξ, Y as well as ψ are in $B_R(0)$ before time $\sigma_R \wedge T$. For the second term we also use the Lipschitz condition to deduce that

$$\|\Phi(\xi_u - \psi_{u+s}(x_0^\varepsilon)) - \Phi(\xi_u - \psi_{u+s}(x_0))\| \leq K_{2R} \|\psi_{u+s}(x_0^\varepsilon) - \psi_{u+s}(x_0)\| .$$

As a consequence of the flow property for ψ this bound approaches 0 as $\varepsilon \to 0$ uniformly w.r.t. $u \in [0, T]$.

By combining these bounds and applying Gronwall's lemma, we find that

$$\begin{aligned} \|\xi_t - Y_t\| \ \leq\ \exp\left\{2K_{2R}t\right\}\Big(& \|\xi_0 - y\| + K_{2R} \int_0^t \|\psi_{u+s}(x_0^\varepsilon) - \psi_{u+s}(x_0)\| \, du \\ & + \int_0^t \left\| b^{\varepsilon, x_0^\varepsilon}(u+s, \xi_u) - \Phi(\xi_u - \psi_{u+s}(x_0^\varepsilon)) \right\| du \Big) \end{aligned} \tag{9.10}$$

for $t \leq \sigma_R$. Since ξ is bounded before σ_R the r.h.s. of this inequality tends to zero by Corollary 9.2.

The exponential equivalence follows from the LDP for Y as follows. Fix $\delta > 0$, and choose $\varepsilon_0 > 0$ such that the r.h.s. of (9.10) is smaller than δ for $\varepsilon \leq \varepsilon_0$. Then $\|\xi_t - Y_t\| > \delta$ implies that at least one of ξ_t or Y_t is not in $B_R(0)$, and if $\xi_t \notin B_R(0)$ then $Y_t \notin B_{R/2}(0)$ if δ is small enough. Thus we can bound the distance of ξ and Y by an exit probability of Y. For $l > 0$ let τ_l denote the diffusion Y's time of first exit from $B_l(0)$. Then, by Proposition 9.3,

$$\begin{aligned} \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}\left(\rho_{0T}(\xi, Y) > \delta\right) \ &\leq\ \limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{P}(\tau_{R/2} \leq T) \\ &\leq\ -\inf\left\{C^s(y, z, t) : \ |z| \geq \tfrac{R}{2}, \ 0 \leq t \leq T\right\}. \end{aligned} \tag{9.11}$$

The latter expression approaches $-\infty$ as $R \to \infty$. □

Theorem 9.4 allows us to deduce two important corollaries. A particular choice of parameters yields an LDP for X, and the ε-dependence of the initial conditions permits us to conclude that the LDP holds uniformly on compact subsets, a fact that is crucial for the proof of an exit law in the following section. The arguments can be found in [17].

Let $\mathbb{P}_{x_0}(X \in \cdot)$ denote the law of the diffusion X starting at $x_0 \in \mathbb{R}^d$.

9.5 Corollary. *Let $L \subset \mathbb{R}^d$ be a compact set.*
For any closed set $F \subset C_{0T}$ we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x_0 \in L} \mathbb{P}_{x_0}(X \in F) \leq -\inf_{x_0 \in L} \inf_{\varphi \in F} I_{0T}^{0,x_0}(\varphi),$$

and for any open set $G \subset C_{0T}$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x_0 \in L} \mathbb{P}_{x_0}(X \in G) \geq -\sup_{x_0 \in L} \inf_{\varphi \in G} I_{0T}^{0,x_0}(\varphi).$$

Proof. Choosing $x_0^\varepsilon = \xi_0^\varepsilon$ and $s = 0$ implies $\xi^\varepsilon = X^\varepsilon$ in Theorem 9.4, which shows that X satisfies an LDP with rate function I_{0T}^{0,x_0}. Furthermore, this LDP allows for ε-dependent initial conditions. This implies the uniformity of the LDP, as pointed out in the proofs of Theorem 5.6.12 and Corollary 5.6.15 in [17]. Indeed, the ε-dependence yields for all $x_0 \in \mathbb{R}^d$

$$\limsup_{\varepsilon \to 0, y \to x_0} \varepsilon \log \mathbb{P}_y(X \in F) \leq -\inf_{\varphi \in F} I_{0T}^{0,x_0}(\varphi),$$

for otherwise one could find sequences $\varepsilon_n > 0$ and $y_n \in \mathbb{R}^d$ such that $\varepsilon_n \to 0$, $y_n \to x_0$ and

$$\limsup_{n \to \infty} \varepsilon_n \log \mathbb{P}_{y_n}(X \in F) > -\inf_{\varphi \in F} I_{0T}^{0,x_0}(\varphi).$$

But this contradicts the LDP.
Now the uniformity of the upper large deviations bound follows exactly as demonstrated in the proof of Corollary 5.6.15 in [17]. The lower bound is treated similarly. □

The next corollary is a consequence of the ε-dependent initial conditions in the LDP for the diffusion ξ.

9.6 Corollary. *Let $L \subset \mathbb{R}^d$ be a compact set.*
For any closed set $F \subset C_{0T}$ we have

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{x_0 \in L} \mathbb{P}(\xi^{s,x_0} \in F) \leq -\inf_{x_0 \in L} \inf_{\varphi \in F} I_{0T}^{s,x_0}(\varphi),$$

and for any open set $G \subset C_{0T}$

$$\liminf_{\varepsilon \to 0} \varepsilon \log \inf_{x_0 \in L} \mathbb{P}(\xi^{s,x_0} \in G) \geq -\sup_{x_0 \in L} \inf_{\varphi \in G} I_{0T}^{s,x_0}(\varphi).$$

9.3 Exponential approximations under stability assumptions

The aim of this section is to exploit the fact that the inhomogeneity of the diffusion $Y^{s,y}$ is weak in the sense that its drift depends on time only through $\psi_{t+s}(x_0)$. If the dynamical system $\dot{\psi} = V(\psi)$ admits an asymptotically stable fixed point x_{stable} that attracts x_0, then the drift of $Y^{s,y}$ becomes almost autonomous for large times, which in turn may be used to estimate large deviations probabilities for $\xi^{s,y}$. We make the following assumption. It will also be in force in Chapter 10, where it will keep us from formulating results on exits from domains with boundaries containing critical points of DV, in particular saddle points in the potential case.

9.7 Assumption.

i) Stability: there exists a stable equilibrium point $x_{\text{stable}} \in \mathbb{R}^d$ of the dynamical system

$$\dot{\psi} = V(\psi).$$

ii) Convexity: the geometry induced by the vector field V is convex, i.e. the condition (8.15) *for V holds globally:*

$$\langle h, DV(x)h\rangle \leq -K_V \tag{9.12}$$

for $h \in \mathbb{R}^d$ s.t. $\|h\| = 1$ and $x \in \mathbb{R}^d$.

Under this assumption it is natural to consider the limiting time homogeneous diffusion $Y^{\infty,y}$ defined by

$$dY_t^\infty = V(Y_t^\infty)dt - \Phi(Y_t^\infty - x_{\text{stable}})dt + \sqrt{\varepsilon}dW_t, \quad Y_0^\infty = y. \tag{9.13}$$

9.8 Lemma. *Let $L \subset \mathbb{R}^d$ be compact, and assume that x_{stable} attracts all $y \in L$, i.e.*

$$\lim_{t\to\infty} \psi_t(y) = x_{\text{stable}} \quad \forall y \in L.$$

Then $Y^{\infty,y}$ is an exponentially good approximation of $Y^{s,y}$, i.e. for any $\delta > 0$ we have

$$\lim_{r\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in L,\, s\geq r} \mathbb{P}\left(\rho_{0T}(Y^{s,y}, Y^{\infty,y}) \geq \delta\right) = -\infty.$$

Proof. We have

$$\begin{aligned}\|Y_t^{s,y} - Y_t^{\infty,y}\| \;\leq\; & \int_0^t \|V(Y_u^{s,y}) - V(Y_u^{\infty,y})\|\, du \\ & + \int_0^t \|\Phi(Y_u^{s,y} - \psi_{s+u}(y)) - \Phi(Y_u^{\infty,y} - x_{\text{stable}})\|\, du.\end{aligned}$$

Let $\sigma_R^{s,y}$ be the first time at which $Y^{s,y}$ or $Y^{\infty,y}$ exits from $B_R(0)$. For $t \leq \sigma_R^{s,y}$, we may use the Lipschitz property of Φ and V, to find a constant $c_R > 0$ s.t.

$$\|Y_t^{s,y} - Y_t^{\infty}\| \leq c_R \int_0^t \|Y_u^{s,y} - Y_u^{\infty}\| du + c_R T \rho_{0T}(\psi_{s+\cdot}(y), x_{\text{stable}}).$$

By assumption the second term on the r.h.s. converges to 0 as $s \to \infty$, uniformly with respect to $y \in L$ since the flow is continuous with respect to the initial data. Hence, by Gronwall's lemma there exists some $r = r(R, \delta) > 0$ such that for $s \geq r$

$$\sup_{y \in L} \sup_{0 \leq t \leq \sigma_R^{s,y}} \|Y_t^{s,y} - Y_t^{\infty}\| < \frac{\delta}{2}.$$

We deduce that

$$\mathbb{P}\left(\rho_{0T}(Y^{s,y}, Y^{\infty}) \geq \delta/2\right) \;\leq\; \mathbb{P}(\tau_{R/2}^y \leq T) \quad \forall s \geq r,\; y \in L,$$

where $\tau_{R/2}^y$ denotes the first exit time of $Y^{\infty,y}$ from $B_{R/2}(0)$. Sending $r, R \to \infty$ and appealing to the uniform LDP for $Y^{\infty,y}$ (Corollary 2.10) finishes the proof, much as the proof of Theorem 9.4. □

This exponential closeness of $Y^{\infty,y}$ and $Y^{s,y}$ carries over to $\xi^{s,y}$ under the aforementioned stability and convexity assumption, which enables us to sharpen the exponential equivalence proved in Theorem 9.4. In order to establish this improvement, we need a preparatory lemma that strengthens Corollary 9.2 to uniform convergence over the whole time axis. This uniformity is of crucial importance for the proof of an exit law in the next section and depends substantially on the strong convexity assumption (9.12).

9.9 Lemma. *We have*

$$\lim_{\varepsilon \to 0} b^{\varepsilon, x_0}(t, x) = \Phi(x - \psi_t(x_0)),$$

uniformly w.r.t. $t \geq 0$ and w.r.t. x and x_0 on compact subsets of $\mathbb{R}^d$.

Proof. Let $f(t) := \mathbb{E}(\|Z_t\|^2)$, where $Z_t = X_t - \psi_t(x_0)$. In the proof of Lemma 9.1 we have seen that

$$\begin{aligned} f'(t) \;&\leq\; 2\,\mathbb{E}\left[\left\langle Z_t, V(Z_t + \psi_t(x_0)) - V(\psi_t(x_0))\right\rangle\right] + \varepsilon d \;\leq\; -2K_V\,\mathbb{E}(\|Z_t\|^2) + \varepsilon d \\ &=\; -2K_V f(t) + \varepsilon d. \end{aligned}$$

This means that $\{t \geq 0 : \; f'(t) < 0\} \supset \{t \geq 0 : \; f(t) > \frac{\varepsilon d}{2K_V}\}$. Recalling that $f(0) = 0$, this allows us to conclude that f is bounded by $\frac{\varepsilon d}{2K_V}$. Now an appeal to the proof of Corollary 9.2 finishes the argument. □

9.10 Proposition. *Let $L \subset \mathbb{R}^d$ be compact, and assume that x_{stable} attracts all $y \in L$. Then $Y^{\infty,y}$ is an exponentially good approximation of $\xi^{s,y}$, i.e. for any $\delta > 0$ we have*

$$\lim_{r\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in L,\, s\geq r} \mathbb{P}\left(\rho_{0T}(\xi^{s,y}, Y^{\infty,y}) \geq \delta\right) = -\infty.$$

Proof. Recall the proof of Theorem 9.4. For $y \in L$ and $s \geq 0$ we have

$$\|\xi_t^{s,y} - Y_t^{s,y}\| \;\leq\; \exp\left\{2K_{2R}t\right\} \int_0^t \|b^{\varepsilon,x_0}(u+s,\xi_u) - \Phi(\xi_u - \psi_{u+s}(x_0))\|\, du \qquad (9.14)$$

for $t \leq \sigma_R^{y,s}$, which denotes the first time that $\xi_t^{s,y}$ or $Y^{s,y}$ exits from $B_R(0)$. By Lemma 9.9, the integrand on the r.h.s. converges to zero as $\varepsilon \to 0$, uniformly w.r.t. $s \geq 0$. Therefore, if we fix $\delta > 0$, we may choose $R = R(\delta)$ sufficiently large and $\varepsilon_0 > 0$ such that for $\varepsilon \leq \varepsilon_0$ and all $s \geq 0$

$$\begin{aligned}\mathbb{P}\left(\rho_{0T}(\xi^{s,y}, Y^{s,y}) > \delta\right) \;&\leq\; \mathbb{P}\left(\tau_{R/2}^{s,y} < T\right)\\ &\leq\; \mathbb{P}\left(\tau_{R/4}^{\infty,y} < T\right) + \mathbb{P}\left(\rho_{0T}(Y^{\infty,y}, Y^{s,y}) > R/4\right).\end{aligned}$$

Here $\tau_l^{s,y}$ denotes the first exit time of the diffusion $Y^{s,y}$ from the ball $B_l(0)$ for $l > 0$, $0 \leq s \leq \infty$. By the uniform LDP for $Y^{\infty,y}$ and Lemma 9.8 the assertion follows. □

The following lemma is the analogue of Lemma 5.7.23 in [17] for the diffusion Y^∞.

9.11 Lemma. *Let $L \subset \mathbb{R}^d$ be compact. For all $\delta > 0$ and $c > 0$ there exists $T > 0$ such that*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \sup_{y\in L} \mathbb{P}\left(\sup_{t\in[0,T]} \|Y_t^{\infty,y} - y\| \geq \delta\right) \;\leq\; -c.$$

Proof. Let φ be the solution of

$$\dot{\varphi}_t = V(\varphi_t) - \Phi(\varphi_t - x_{\text{stable}}), \quad \varphi_0 = y.$$

Since $\sup_{t\in[0,T]} \|\varphi_t - y\| < \delta/2$ for sufficiently small T, uniformly w.r.t. $y \in L$ due to the continuity of the flow φ, we may find $T_0 > 0$ such that

$$\mathbb{P}\left(\sup_{t\in[0,T]} \|Y_t^{\infty,y} - y\| \geq \delta\right) \;\leq\; \mathbb{P}\left(\sup_{t\in[0,T]} \|Y_t^{\infty,y} - \varphi_t\| \geq \delta/2\right)$$

for $T \leq T_0$ and $y \in L$. Let us estimate the distance of Y^∞ and φ. We have

$$\begin{aligned}\|Y_t^\infty - \varphi_t\|^2 =& 2\int_0^t \langle Y_u^\infty - \varphi_u, V(Y_u^\infty) - V(\varphi_u)\rangle\, du\\ &- 2\int_0^t \langle Y_u^\infty - \varphi_u, \Phi(Y_u^\infty - x_{\text{stable}}) - \Phi(\varphi_u - x_{\text{stable}})\rangle\, du\\ &+ 2\sqrt{\varepsilon}\int_0^t \langle Y_u^\infty - \varphi_u, dW_u\rangle + \varepsilon dt.\end{aligned}$$

The integrands of the first and the second integral on the r.h.s. of this equation are negative resp. positive due to the dissipativity properties of V and Φ. Using a localization argument, we see that $Z := Y^\infty - \varphi$ is square integrable, hence the stochastic integral is a martingale and $\mathbb{E}\left[\, \|Z_t\|^2 \,\right] \le \varepsilon dt$.
Now the problem of deriving a bound for $\sup_{t\in[0,T]} \|Z_t\|$ amounts to estimating the running maximum of the martingale

$$M_t = \sum_{i=1}^d M_t^i,$$

where $M_t^i = \int_0^t Z_u^i dW_u^i$ and Z^i, W^i denote the components of Z and W, respectively. Each M^i may be written as a time-changed Brownian motion, i.e. there exist one-dimensional Brownian motions B^i and random time changes $\tau^i(t)$, $i = 1, \ldots, d$, $t \ge 0$, such that each $\tau^i(t)$, $t \ge 0$, is a stopping time for B^i and

$$M_t = B^i_{\tau^i(t)}.$$

Moreover, $\tau^i(t) = \langle M^i \rangle_t \le \int_0^t \|Z_u\|^2 du$, which implies for $\alpha > 0$

$$\mathbb{P}\left(\tau^i(t) \ge \alpha T\right) \;\le\; \frac{\mathbb{E}[\tau^i(t)]}{\alpha T} \;\le\; \frac{1}{\alpha T}\int_0^t \mathbb{E}[\|Z_u\|^2]du \;\le\; \frac{\varepsilon dt}{2\alpha T}.$$

Thus, using the exponential inequality for Brownian motion, we obtain for $i = 1, \ldots, d$ and $\alpha, \delta > 0$

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} \left|M_t^i\right| \ge \delta\right) \;&\le\; \mathbb{P}\left(\sup_{t\in[0,\alpha T]} \left|B_t^i\right| \ge \delta\right) + \mathbb{P}\left(\tau^i(T) \ge \alpha T\right) \\
&\le\; 2\exp\left\{-\frac{\delta^2}{2\alpha T}\right\} + \frac{\varepsilon d}{2\alpha}.
\end{aligned}$$

This yields the following estimate for Z. We have for $\varepsilon < \frac{\delta}{2dT}$

$$\begin{aligned}
\mathbb{P}\left(\sup_{t\in[0,T]} \|Z_t\| \ge \delta\right) \;&\le\; \mathbb{P}\left(2\sqrt{\varepsilon}\sup_{t\in[0,T]} \|M_t\| \ge \frac{\delta}{2}\right) \\
&\le\; \sum_{i=1}^d \mathbb{P}\left(\sup_{t\in[0,T]} \left\|M_t^i\right\| \ge \frac{\delta}{4\sqrt{d\varepsilon}}\right) \\
&\le\; 2d\, \exp\left\{-\,\frac{\delta^2}{32\,\alpha T d\varepsilon}\right\} + \frac{\varepsilon d^2}{2\alpha},
\end{aligned}$$

indepently of $y \in \mathbb{R}^d$. This finally leads to

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left(\sup_{t\in[0,T]} \|Z_t\| \ge \delta\right) \;\le\; -\frac{\delta^2}{32\,\alpha T d},$$

which becomes less than $-c$ for sufficiently small $\alpha > 0$. $\square$

As an immediate consequence, we obtain the following result for $\xi^{s,y}$ by combining Proposition 9.10 and Lemma 9.11.

9.12 Lemma. *Let $L \subset \mathbb{R}^d$ be compact, and assume that x_{stable} attracts all $y \in L$. For all $\delta > 0$ and $c > 0$ there exist $T > 0$ and $r_0 > 0$ such that*

$$\limsup_{\varepsilon \to 0} \varepsilon \log \sup_{y \in L,\, s \geq r_0} \mathbb{P}\Big(\sup_{t \in [0,T]} \|\xi_t^{s,y} - y\| \geq \delta \Big) \;\leq\; -c.$$

Chapter 10

The exit problem

In this chapter, we shall address the problem of *noise induced exit* for the self-stabilizing diffusion, and derive the precise large deviations asymptotics of exit times from domains of attraction of the underlying deterministic system. More precisely, under convexity assumptions on the geometry of V, we shall extend Kramers' law for time homogeneous diffusions (Theorem 3.1) to our class of self-stabilizing diffusions. Intuitively, exit times should increase compared to the classical case due to self-stabilization and the inertia it entails. We shall show that this is indeed the case. Our approach follows the presentation in [17].

Let D be a bounded domain in $\mathbb{R}^d$ in which X^ε starts, i.e. $x_0 \in D$, and denote by

$$\tau_D^\varepsilon = \inf\{t > 0 : \; X_t^\varepsilon \in \partial D\}$$

the first exit time from D. We make the following stability assumptions about D.

10.1 Assumption.

i) The unique equilibrium point in D of the dynamical system

$$\dot{\psi}_t = V(\psi_t) \tag{10.1}$$

is stable and given by $x_{\text{stable}} \in D$. As before, $\psi_t(x_0)$ denotes the solution starting in x_0. We assume that $\lim_{t\to\infty} \psi_t(x_0) = x_{\text{stable}}$.

ii) The solutions of

$$\dot{\varphi}_t = V(\varphi_t) - \Phi(\varphi_t - x_{\text{stable}}) \tag{10.2}$$

satisfy

$$\varphi_0 \in D \quad \Longrightarrow \quad \varphi_t \in D \;\; \forall t > 0 \;\;\text{and}\;\; \lim_{t\to\infty} \varphi_t = x_{\text{stable}},$$

and all trajectories starting at the boundary ∂D converge to the stable point x_{stable}.

The description of the exponential rate for the exit time of Itô diffusions with homogeneous coefficients was first proved by Freidlin and Wentzell by exploiting the strong Markov property. The self-stabilizing diffusion X^ε is also Markovian, but it is inhomogeneous, which makes a direct application of the Markov property difficult. However, the inhomogeneity is weak under the stability Assumption 10.1. It implies that the law of X_t^ε converges as time tends to infinity, and large deviations probabilites for X^ε may be approximated by those of Y^∞ in the sense of Proposition 9.10. Since Y^∞ is defined in terms of an autonomous SDE, its exit behavior is accessible through classical results. The rate function that describes the LDP for Y^∞ is given by

$$I_{0T}^{\infty,y}(\varphi) = \begin{cases} \frac{1}{2}\int_0^T \left\|\dot{\varphi}_t - V(\varphi_t) + \Phi(\varphi_t - x_{\text{stable}})\right\|^2 \, dt, & \text{if } \varphi \in H_{0T}^y, \\ \infty, & \text{otherwise .} \end{cases} \tag{10.3}$$

The corresponding cost function and quasi potential are defined in an obvious way and denoted by C^∞ and Q^∞, respectively. The minimal energy required to connect the stable equilibrium point x_{stable} to the boundary of the domain is assumed to be finite, i.e.

$$\overline{Q}_\infty := \inf_{z\in\partial D} Q^\infty(x_{\text{stable}}, z) < \infty.$$

The following two theorems state our main result about the exponential rate of the exit time and the exit location.

10.2 Theorem. *For all $x_0 \in D$ and all $\eta > 0$, we have*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \left\{1 - \mathbb{P}_{x_0}(e^{(\overline{Q}_\infty-\eta)/\varepsilon} < \tau_D^\varepsilon < e^{(\overline{Q}_\infty+\eta)/\varepsilon})\right\} \leq -\eta/2, \tag{10.4}$$

and

$$\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{x_0}(\tau_D^\varepsilon) = \overline{Q}_\infty. \tag{10.5}$$

10.3 Theorem. *If $N \subset \partial D$ is a closed set satisfying* $\inf_{z\in N} Q^\infty(x_{\text{stable}}, z) > \overline{Q}_\infty$*, then it does not see the exit point: for any $x_0 \in D$*

$$\lim_{\varepsilon\to 0} \mathbb{P}_{x_0}(X_{\tau_D}^\varepsilon \in N) = 0.$$

The rest of this chapter is devoted to the proof of these two theorems. In the following chapter, these results are illustrated by examples which show that the attraction part of the drift term in a diffusion may completely change the behavior of the paths, i.e. the self-stabilizing diffusion stays in the domain for a longer time than the classical one, and it typically exits at a different place.

10.1 Enlargement of the domain

The self-stabilizing diffusion lives in the bounded domain D which is assumed to fulfill the previously stated stability conditions. In order to derive upper and lower bounds of exit probabilities, we need to construct an enlargement of D that still enjoys the stability properties of Assumption 10.1 ii). This is possible because the family of solutions to the dynamical system (10.2) defines a continuous flow.

For $\delta > 0$ we denote by $D^\delta := \{y \in \mathbb{R}^d : \operatorname{dist}(y, D) < \delta\}$ the open δ-neighborhood of D. The flow φ is continuous, hence uniformly continuous on $\overline{D}$ due to boundedness of D, and since the vector field is locally Lipschitz. Hence, if δ is small enough, the trajectories $\varphi_t(y)$ converge to x_{stable} for $y \in D^\delta$, i.e. for each neighborhood $\mathcal{V} \subset D$ of x_{stable} there exists some $T > 0$ such that for $y \in D^\delta$ we have $\varphi_t(y) \in \mathcal{V}$ for all $t \geq T$. Moreover, the joint continuity of the flow implies that, if we fix $c > 0$, we may choose $\delta = \delta(c) > 0$ such that

$$\sup \Big\{ \operatorname{dist}(\varphi_t(y), D) : \ t \in [0, T], \ y \in D^\delta \Big\} < c.$$

Let

$$\mathcal{O}^\delta = \Big\{ y \in \mathbb{R}^d : \sup_{t \in [0,T]} \operatorname{dist}(\varphi_t(y), D) < c, \ \varphi_T(y) \in \mathcal{V} \Big\}.$$

Then $\mathcal{O}^\delta$ is a bounded open set which contains D^δ and satisfies Assumption 10.1 ii). Indeed, if δ is small enough, the boundary of $\mathcal{O}^\delta$ is not a characteristic boundary, and $\cap_{\delta > 0} \mathcal{O}^\delta = D$.

10.2 Proof of the upper bound for the exit time

For the proof of the two main results, we successively proceed in several steps and establish a series of preparatory estimates that shall be combined afterwards. In this section, we concentrate on the upper bound for the exit time from D, and establish inequalities for the probability of exceeding this bound and for the mean exit time.

In the sequel, we denote by $\mathbb{P}_{s,y}$ the law of the diffusion $\xi^{s,y}$, defined by (9.3). Recall that by the results of the previous chapter, $\xi^{s,y}$ satisfies a large deviations principle with rate function $I^{s,y}$. The following continuity property of the associated cost function is the analogue of Lemma 5.7.8 in [17] for this inhomogeneous diffusion. The proof is omitted.

10.4 Lemma. *For any $\delta > 0$ and $s \in [0, \infty)$, there exists $\varrho > 0$ such that*

$$\sup_{x,y \in B_\varrho(x_{\text{stable}})} \inf_{t \in [0,1]} C^s(x, y, t) < \delta \tag{10.6}$$

and

$$\sup_{(x,y)\in\Gamma} \inf_{t\in[0,1]} C^s(x,y,t) < \delta, \tag{10.7}$$

where $\Gamma = \{(x,y) : \inf_{z\in\partial D}(\|y-z\| + \|x-z\|) \le \varrho\}$.

Let us now present two preliminary lemmas on exit times of $\xi^{s,y}$. In slight abuse of notation, we denote exit times of $\xi^{s,y}$ also by τ_D^ε, which could formally be justified by assuming to look solely at the coordinate process on path space and switching between measures instead of processes. On the other hand, this notation is convenient when having in mind that $\xi^{s,y}$ describes the law of X^ε restarted at time s, and that X^ε may be recovered from $\xi^{s,y}$ for certain parameters.

10.5 Lemma. *For any $\eta > 0$ and $\varrho > 0$ small enough, there exist $T_0 > 0$, $s_0 > 0$ and $\varepsilon_0 > 0$ such that*

$$\inf_{y\in B_\varrho(x_{\text{stable}})} \mathbb{P}_{s,y}\left(\tau_D^\varepsilon \le T_0\right) \;\ge\; e^{-(\overline{Q}_\infty+\eta)/\varepsilon}$$

for all $\varepsilon \le \varepsilon_0$ and $s \ge s_0$.

Proof. Let ϱ be given according to Lemma 10.4. The corresponding result for the time homogeneous diffusion $Y^{\infty,y}$ is well known (see [17], Lemma 5.7.18, resp. Lemma B.1), and will be carried over to $\xi^{s,y}$ using the exponential approximation of Proposition 9.10.
Let $\mathbb{P}_{\infty,y}$ denote the law of $Y^{\infty,y}$. The drift of $Y^{\infty,y}$ is locally Lipschitz by the assumptions on V and Φ, and we may assume w.l.o.g. that it is even globally Lipschitz. Otherwise we change the drift outside a large domain containing D.
If $\delta > 0$ is small enough such that the enlarged domain $\mathcal{O}^\delta$ satisfies Assumption 10.1 ii), Lemma B.1 implies the existence of ε_1 and T_0 such that

$$\inf_{y\in B_\varrho(x_{\text{stable}})} \mathbb{P}_{\infty,y}(\tau_{\mathcal{O}^\delta}^\varepsilon \le T_0) \ge e^{-(\overline{Q}_\infty^\delta+\eta/3)/\varepsilon} \text{ for all } \varepsilon \le \varepsilon_1. \tag{10.8}$$

Here $\overline{Q}_\infty^\delta$ denotes the minimal energy

$$\overline{Q}_\infty^\delta = \inf_{z\in\partial\mathcal{O}^\delta} Q^\infty(x_{\text{stable}}, z).$$

The continuity of the cost function carries over to the quasi-potential, i.e. there exists some $\delta_0 > 0$ such that $|\overline{Q}_\infty^\delta - \overline{Q}_\infty| \le \eta/3$ for $\delta \le \delta_0$.
Now let us link the exit probabilities of $Y^{\infty,y}$ and $\xi^{s,y}$. We have for $s \ge 0$

$$\begin{aligned} \mathbb{P}_{s,y}(\tau_D^\varepsilon \le T_0) \;&\ge\; \mathbb{P}(\{\xi^{s,y} \text{ exits from } D \text{ before } T_0\} \cap \{\rho_{0,T_0}(\xi^{s,y}, Y^{\infty,y}) \le \delta\}) \\ &\ge\; \mathbb{P}_{\infty,y}(\tau_{D^\delta}^\varepsilon \le T_0) - \mathbb{P}(\rho_{0,T_0}(\xi^{s,y}, Y^\infty) \ge \delta). \end{aligned} \tag{10.9}$$

Moreover, by the exponential approximation we may find $\varepsilon_2 > 0$ and $s_0 > 0$ such that

$$\sup_{y \in B_\varrho(x_{\text{stable}})} \mathbb{P}(\rho_{0,T_0}(\xi^{s,y}, Y^\infty) \geq \delta) \leq e^{-(\overline{Q}_\infty^\delta + \eta/2)/\varepsilon} \quad \forall s \geq s_0,\ \varepsilon \leq \varepsilon_2.$$

Since $D^\delta \subset \mathcal{O}^\delta$, we deduce that for $\varepsilon \leq \varepsilon_0 = \varepsilon_1 \wedge \varepsilon_2$ and $s \geq s_0$

$$\inf_{y \in B_\varrho(x_{\text{stable}})} \mathbb{P}_{s,y}(\tau_D^\varepsilon \leq T_0) \geq e^{-(\overline{Q}_\infty^\delta + \eta/3)/\varepsilon} - e^{-(\overline{Q}_\infty^\delta + \eta/2)/\varepsilon} \geq e^{-(\overline{Q}_\infty^\delta + \eta)/\varepsilon}. \qquad \square$$

By similar arguments, we prove the exponential smallness of the probability of too long exit times. Let $\Sigma_\varrho = \inf\{t \geq 0 :\ \xi_t^{s,y} \in B_\varrho(x_{\text{stable}}) \cup \partial D\}$, where ϱ is small enough such that $B_\varrho(x_{\text{stable}})$ is contained in the domain D.

10.6 Lemma. *For any $\varrho > 0$ and $K > 0$ there exist $\varepsilon_0 > 0$, $T_1 > 0$ and $r > 0$ such that*

$$\sup_{y \in D,\, s \geq r} \mathbb{P}_{s,y}(\Sigma_\varrho > t) \leq e^{-K/\varepsilon} \quad \forall t \geq T_1.$$

Proof. As before, we use the fact that a similar result is already known for $Y^{\infty,y}$. For $\delta > 0$ small enough, let

$$\Sigma_\varrho^\delta = \inf\{t \geq 0 :\ Y_t^\infty \in B_{\varrho-\delta}(x_{\text{stable}}) \cup \partial\mathcal{O}^\delta\}.$$

By Lemma 5.7.19 in [17] (Lemma B.2), there exist $T_1 > 0$ and $\varepsilon_1 > 0$ such that

$$\sup_{y \in D} \mathbb{P}_{\infty,y}(\Sigma_\varrho^\delta > t) \leq e^{-K/\varepsilon} \quad \forall t \geq T_1 \quad \varepsilon \leq \varepsilon_1.$$

Now the assertion follows from

$$\sup_{y \in D} \mathbb{P}_{s,y}(\Sigma_\varrho > T_1) \leq \sup_{y \in D} \mathbb{P}_{\infty,y}(\Sigma_\varrho^\delta > T_1) + \sup_{y \in D} \mathbb{P}(\rho_{0,T_1}(\xi^{s,y}, Y^{\infty,y}) > \delta),$$

since the last term is exponentially negligible by Proposition 9.10. $\square$

The previous two lemmas contain the essential large deviations bounds required for the proof of the following upper bound for the exit time of X^ε.

10.7 Proposition. *For all $x_0 \in D$ and $\eta > 0$ we have*

$$\limsup_{\varepsilon \to 0} \varepsilon \log\ \mathbb{P}_{x_0}\left(\tau_D^\varepsilon \geq e^{(\overline{Q}_\infty + \eta)/\varepsilon}\right) \ \leq\ -\eta/2, \tag{10.10}$$

$$\limsup_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_0}\left[\tau_D^\varepsilon\right] \ \leq\ \overline{Q}_\infty. \tag{10.11}$$

Proof. The proof consists of a careful modification of the arguments used in Theorem 5.7.11 in [17]. By Lemma 10.5 and Lemma 10.6, there exist $\widetilde{T} = T_0 + T_1 > 0$, $\varepsilon_0 > 0$ and $r_0 > 0$ such that for $T \geq \widetilde{T}$, $\varepsilon \leq \varepsilon_0$ and $r \geq r_0$ we have

$$q_T^r := \inf_{y \in D} \mathbb{P}_{r,\,y}(\tau_D^\varepsilon \leq T) \;\geq\; \inf_{y \in D} \mathbb{P}_{r,\,y}\left(\Sigma_\varrho \leq T_1\right) \inf_{y \in B_\varrho(x_{\text{stable}}),\, s \geq r} \mathbb{P}_{s,\,y}\left(\tau_D^\varepsilon \leq T_0\right) \tag{10.12}$$

$$\geq \exp\left\{ - \frac{\overline{Q}_\infty + \eta/2}{\varepsilon} \right\} =: q_T^\infty .$$

Moreover, by the Markov property of $\xi^{s,y}$, we see that for $k \in \mathbb{N}$

$$\begin{aligned}
\mathbb{P}_{x_0}\left(\tau_D^\varepsilon > 2(k+1)T\right) \;&=\; \left[1 - \mathbb{P}_{x_0}\left(\tau_D^\varepsilon \leq 2(k+1)T \mid \tau_D^\varepsilon > 2kT\right)\right] \mathbb{P}_{x_0}\left(\tau_D^\varepsilon > 2kT\right) \\
&\leq\; \left[1 - \inf_{y \in D} \mathbb{P}_{2kT,y}\left(\tau_D^\varepsilon \leq 2T\right)\right] \mathbb{P}_{x_0}\left(\tau_D^\varepsilon > 2kT\right) \\
&\leq\; \left(1 - q_{2T}^{2kT}\right) \mathbb{P}_{x_0}\left(\tau_D^\varepsilon > 2kT\right),
\end{aligned}$$

which by induction yields

$$\mathbb{P}_{x_0}\left(\tau_D^\varepsilon > 2kT\right) \;\leq\; \prod_{i=0}^{k-1} \left(1 - q_{2T}^{2iT}\right). \tag{10.13}$$

Let us estimate each term of the product separately. We have

$$\begin{aligned}
1 - q_{2T}^{2iT} \;&=\; \sup_{y \in D} \mathbb{P}_{2iT,y}\left(\tau_D^\varepsilon > 2T\right) \;\leq\; \sup_{y \in D} \mathbb{P}_{2iT,y}\left(\tau_D^\varepsilon > T\right) \sup_{y \in D} \mathbb{P}_{(2i+1)T,y}\left(\tau_D^\varepsilon > T\right) \\
&\leq\; \sup_{y \in D} \mathbb{P}_{(2i+1)T,y}\left(\tau_D^\varepsilon > T\right).
\end{aligned}$$

By choosing T large enough, we may replace the product in (10.13) by a power. Indeed, for $T > \max(\widetilde{T}, r_0)$ we have $(2i+1)T \geq r_0$ for all $i \in \mathbb{N}$, which by (10.12) results in the uniform upper bound

$$1 - q_{2T}^{2iT} \;\leq\; 1 - q_T^{(2i+1)T} \;\leq\; 1 - q_T^\infty .$$

By plugging this into (10.13), we obtain a 'geometric' upper bound for the expected exit time, namely

$$\begin{aligned}
\mathbb{E}_{x_0}\left[\tau_D^\varepsilon\right] \;&\leq\; 2T\Big[1 + \sum_{k=1}^\infty \sup_{y \in D} \mathbb{P}_{x_0}\left(\tau_D^\varepsilon \geq 2kT\right)\Big] \;\leq\; 2T\Big[1 + \sum_{k=1}^\infty \prod_{i=0}^{k-1} \left(1 - q_{2T}^{2iT}\right)\Big] \\
&\leq\; 2T\Big[1 + \sum_{k=1}^\infty \left(1 - q_T^\infty\right)^k\Big] = \frac{2T}{q_T^\infty}.
\end{aligned}$$

This proves the claimed asymptotics of the expected exit time. Furthermore, an application of Chebychev's inequality shows that

$$\mathbb{P}_{x_0}\left(\tau_D^\varepsilon \geq e^{(\overline{Q}_\infty + \eta)/\varepsilon}\right) \;\leq\; \frac{\mathbb{E}_{x_0}\left[\tau_D^\varepsilon\right]}{e^{(\overline{Q}_\infty + \eta)/\varepsilon}} \;\leq\; 2T\, \frac{e^{-(\overline{Q}_\infty + \eta)/\varepsilon}}{q_T^\infty} \;=\; 2T\, e^{-\eta/2\varepsilon},$$

which is the asserted upper bound of the exit probability. □

10.3 Proof of the lower bound for the exit time

In order to establish the lower bound of the exit time, we prove a preliminary lemma which estimates the probability to exit from the domain $D \backslash B_\varrho(x_{\text{stable}})$ at the boundary of D. This probability is seen to be exponentially small since the diffusion is attracted to the stable equilibrium point. Let us denote by S_ϱ the boundary of $B_\varrho(x_{\text{stable}})$, and recall the definition of the stopping time Σ_ϱ.

10.8 Lemma. *For any closed set $N \subset \partial D$ and $\eta > 0$, there exist $\varepsilon_0 > 0$, $\varrho_0 > 0$ and $r_0 > 0$ such that*

$$\varepsilon \log \sup_{y \in S_{2\varrho},\, s \geq r} \mathbb{P}\left(\xi^{s,y}_{\Sigma_\varrho} \in N\right) \;\leq\; -\inf_{z \in N} Q^\infty(x_{\text{stable}}, z) + \eta$$

for all $\varepsilon \leq \varepsilon_0$, $r \geq r_0$ and $\varrho \leq \varrho_0$.

Proof. For $\delta > 0$ we define a subset $\mathcal{S}^\delta$ of D^δ by setting

$$\mathcal{S}^\delta := D^\delta \setminus \{y \in \mathbb{R}^d : \; \mathrm{dist}(y, N) < \delta\}.$$

Furthermore, let

$$\mathcal{N}^\delta := \partial\mathcal{S}^\delta \cap \{y \in \mathbb{R}^d : \; \mathrm{dist}(y, N) \leq \delta\}.$$

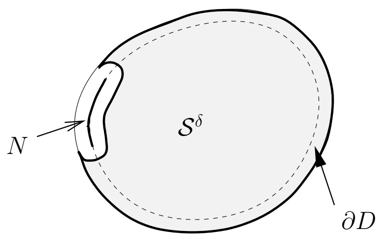

Figure 10.1: Definition of $\mathcal{S}^\delta$

$\mathcal{S}^\delta$ contains the stable equilibrium point x_{stable}, and as such it is unique in $\mathcal{S}^\delta$ if δ is small enough. Since an exit of the limiting diffusion Y^∞ from the domain $\mathcal{O}^\delta$ defined in Section 10.1 always requires an exit from $\mathcal{S}^\delta$, we may apply Lemma 5.7.19 in [17] (resp. Lemma B.3), to find $\varepsilon_1 > 0$ and $\varrho_1 > 0$ such that

$$\varepsilon \log \sup_{y \in S_{2\varrho}} \mathbb{P}_{\infty, y}\left(Y^\infty_{\Sigma^\delta_\varrho} \in \mathcal{N}^\delta\right) \leq -\inf_{z \in \mathcal{N}^\delta} Q^\infty(x_{\text{stable}}, z) + \frac{\eta}{2}$$

for $\varepsilon \leq \varepsilon_1$ and $\varrho \leq \varrho_1$, where Σ^δ_ϱ denotes the first exit time from the domain $\mathcal{S}^\delta \setminus B_\varrho(x_{\text{stable}})$. By the continuity of the quasi-potential we have

$$-\inf_{z \in \mathcal{N}^\delta} Q^\infty(x_{\text{stable}}, z) + \frac{\eta}{2} \;\leq\; -\inf_{z \in N} Q^\infty(x_{\text{stable}}, z) + \eta$$

if $\delta > 0$ is small enough. Therefore, it is sufficient to link the result about the limiting diffusion to the corresponding statement dealing with $\xi^{s,y}$. By Lemma 10.6, we can find $T_1 > 0$, $\varepsilon_1 > 0$ and $r_1 > 0$ such that

$$\varepsilon \log \sup_{y \in S_{2\varrho},\, s \geq r} \mathbb{P}_{s,y}\left(\Sigma_\varrho \geq T_1\right) \;\leq\; -\inf_{z \in N} Q^\infty(x_{\text{stable}}, z) + \frac{\eta}{2} \qquad \forall \varepsilon \leq \varepsilon_1,\; r \geq r_1. \tag{10.14}$$

If $\Sigma_\varrho \le T_1$ and $\rho_{0,T_1}(\xi^{s,y}, Y^\infty) \le \delta$, then $\{\xi^{s,y}_{\Sigma_\varrho} \in N\}$ is contained in $\{Y^\infty_{\Sigma^\delta_\varrho} \in \mathcal{N}^\delta\}$. Thus,

$$\begin{aligned}\mathbb{P}\left(\xi^{s,y}_{\Sigma_\varrho} \in N\right) \;&\le\; \mathbb{P}\left(\xi^{s,y}_{\Sigma_\varrho} \in N,\, \Sigma_\varrho < T_1\right) + \mathbb{P}_{s,y}(\Sigma_\varrho \ge T_1)\\ &\le\; \mathbb{P}\left(Y^{\infty,y}_{\Sigma^\delta_\varrho} \in \mathcal{N}^\delta\right) + \mathbb{P}\left(\rho_{0,T_1}(\xi^{s,y}, Y^{\infty,y}) \ge \delta\right) + \mathbb{P}_{s,y}(\Sigma_\varrho \ge T_1).\end{aligned}$$

By (10.14) and Proposition 9.10, the logarithmic asymptotics of the sum on the r.h.s. is dominated by the first term, i.e. the lemma is established. □

We are now in a position to establish the lower bound for the exit time which complements Proposition 10.7 and completes the proof of Theorem 10.2.

10.9 Proposition. *There exists $\eta_0 > 0$ such that for any $\eta \le \eta_0$*

$$\limsup_{\varepsilon\to 0} \varepsilon \log \mathbb{P}_{x_0}\left[\tau^\varepsilon_D < e^{(\overline{Q}_\infty - \eta)/\varepsilon}\right] \;\le\; -\eta/2 \tag{10.15}$$

and

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_{x_0}\left[\tau^\varepsilon_D\right] \;\ge\; \overline{Q}_\infty. \tag{10.16}$$

Proof. In a first step we apply Lemma 10.8 and Lemma 9.12. We find $r_0 > 0$, $T > 0$ and $\varepsilon_0 > 0$ such that for $\varepsilon \le \varepsilon_0$

$$\sup_{y\in S_{2\varrho},\, s\ge r_0} \mathbb{P}\left(\xi^{s,y}_{\Sigma_\varrho} \in \partial D\right) \;\le\; e^{-(\overline{Q}_\infty - \eta/2)/\varepsilon},$$

$$\sup_{y\in D,\, s\ge r_0} \mathbb{P}\left(\sup_{0\le t\le T} \|\xi^{s,y}_t - y\| \ge \varrho\right) \;\le\; e^{-(\overline{Q}_\infty - \eta/2)/\varepsilon}.$$

In the sequel, we shall proceed as follows. Firstly, we wait for a large period of time r_1 until the diffusion becomes ‘sufficiently homogeneous’, which is possible thanks to the stability assumption. Since x_{stable} attracts all solutions of the deterministic system, we may find $r_1 \ge r_0$ such that $\psi_r(x_0) \in B_\varrho(x_{\text{stable}})$ for $r \ge r_1$. Secondly, after time r_1, we employ the usual arguments for homogeneous diffusions. Following [17], we recursively define two sequences of stopping times that shall serve to track the diffusion’s excursions between the small ball $B_\varrho(x_{\text{stable}})$ around the equilibrium point and the larger sphere $S_{2\varrho} = \partial B_{2\varrho}(x_{\text{stable}})$, before it finally exits from the domain D.

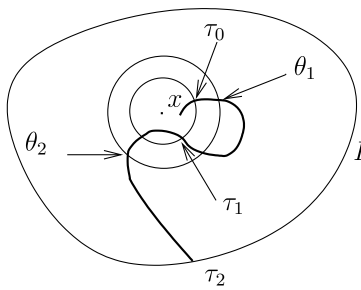

Figure 10.2: Excursions

Set $\vartheta_0 = r_1$, and for $m \ge 0$ let

$$\tau_m = \inf\{t \ge \vartheta_m :\; X^\varepsilon_t \in B_\varrho \cup \partial D\},$$

and

$$\vartheta_{m+1} = \inf\{t > \tau_m : \ X^\varepsilon_t \in S_{2\varrho}\}.$$

Let us decompose the event $\{\tau^\varepsilon_D \leq kT + r_1\}$. We have

$$\begin{aligned}\mathbb{P}_{x_0}\left(\tau^\varepsilon_D \leq kT + r_1\right) \ &\leq \ \mathbb{P}_{x_0}\left(\{\tau^\varepsilon_D \leq r_1\} \cup \{X^\varepsilon_{r_1} \notin B_{2\varrho}(x_{\text{stable}})\}\right) \\ &\quad + \sup_{y\in S_{2\varrho},\, s\geq r_1} \mathbb{P}_{s,\,y}(\tau^\varepsilon_D \leq kT).\end{aligned} \tag{10.17}$$

The first probability on the r.h.s. of this inequality tends to 0 as $\varepsilon \to 0$. Indeed, by the large deviations principle for X^ε on the time interval $[0, r_1]$, there exist $\eta_0 > 0$ and $\varepsilon_2 > 0$ such that

$$\varepsilon \log \mathbb{P}_{x_0}\left(\{\tau^\varepsilon_D \leq r_1\} \cup \{X^\varepsilon_{r_1} \notin B_{2\varrho}(x_{\text{stable}})\}\right) \ \leq \ -\eta/2$$

for $\varepsilon \leq \varepsilon_2$ and $\eta \leq \eta_0$. For the second term in (10.17), we can observe two different cases: either the diffusion exits from D during the first k exits from $D \setminus B_\varrho(x_{\text{stable}})$, or the minimal time spent between two consecutive exits is smaller than T. This reasoning leads to the bound

$$\mathbb{P}_{s,\,y}(\tau^\varepsilon_D \leq kT) \ \leq \ \sum_{m=0}^{k} \mathbb{P}_{s,\,y}\left(\tau^\varepsilon_D = \tau_m\right) + \mathbb{P}_{s,\,y}\left(\min_{1\leq m\leq k}(\vartheta_m - \tau_{m-1}) \leq T\right).$$

Let us now link these events to the probabilities presented at the beginning of the proof. We have

$$\sup_{y\in S_{2\varrho},\, s\geq r_1} \mathbb{P}_{s,\,y}(\tau^\varepsilon_D = \tau_m) \leq \sup_{y\in S_{2\varrho},\, s\geq r_0} \mathbb{P}_{s,\,y}(\xi^{s,\,y}_{\Sigma_\varrho} \in \partial D),$$

and

$$\sup_{y\in S_{2\varrho},\, s\geq r_1} \mathbb{P}_{s,\,y}\left((\vartheta_m - \tau_{m-1}) \leq T\right) \leq \sup_{y\in S_{2\varrho},\, s\geq r_0} \mathbb{P}_{s,\,y}\left(\sup_{0\leq t\leq T} \|\xi^{s,\,y}_t - y\| \geq \varrho\right),$$

which yields the bound

$$\sup_{y\in S_{2\varrho},\, s\geq r_1} \mathbb{P}_{s,\,y}(\tau^\varepsilon_D \leq kT) \leq (2k+1)e^{-(\overline{Q}_\infty - \eta/2)/\varepsilon}.$$

Thus, by choosing $k = \lfloor (e^{(\overline{Q}_\infty - \eta)/\varepsilon} - r_1)/T \rfloor + 1$, we obtain from (10.17)

$$\mathbb{P}_{x_0}(\tau^\varepsilon_D \leq e^{(\overline{Q}_\infty - \eta)/\varepsilon}) \ \leq \ e^{-\eta/2\varepsilon} + 5T^{-1}e^{-\eta/2\varepsilon},$$

i.e. (10.15) holds. Moreover, by using Chebychev's inequality, we obtain the claimed lower bound for the expected exit time. Indeed, we have

$$\mathbb{E}_{x_0}(\tau^\varepsilon_D) \geq e^{(\overline{Q}_\infty - \eta)/\varepsilon}(1 - \mathbb{P}_{x_0}(\tau^\varepsilon_D \leq e^{(\overline{Q}_\infty - \eta)/\varepsilon})) \geq e^{(\overline{Q}_\infty - \eta)/\varepsilon}(1 - (1 + 5T^{-1})e^{-\eta/2\varepsilon}),$$

which establishes (10.16). □

We end this chapter with the proof of Theorem 10.3 about the exit location.

Proof of Theorem 10.3. We use arguments similar to the ones of the preceding proof. Let

$$\overline{Q}_\infty(N) = \inf_{z\in N} Q^\infty(x_{\text{stable}}, z),$$

and assume w.l.o.g. that $\overline{Q}_\infty < \overline{Q}_\infty(N) < \infty$. Otherwise, we may replace $\overline{Q}_\infty(N)$ in the following by some constant larger than $\overline{Q}_\infty$. As in the preceding proof, we may choose $T > 0$, $r_0 > 0$ and $\varepsilon_0 > 0$ such that

$$\sup_{y\in S_{2\varrho},\, s\geq r_0} \mathbb{P}_{s,y}\left(\xi^{s,y}_{\Sigma_\varrho} \in \partial N\right) \leq e^{-(\overline{Q}_\infty(N)-\eta/2)/\varepsilon} \quad \forall\varepsilon \leq \varepsilon_0,$$

$$\sup_{y\in D,\, s\geq r_0} \mathbb{P}_{s,y}\left(\sup_{0\leq t\leq T} \|\xi^{s,y}_t - y\| \geq \varrho\right) \leq e^{-(\overline{Q}_\infty(N)-\eta/2)/\varepsilon} \quad \forall\varepsilon \leq \varepsilon_0.$$

It suffices to study the event $A = \{\tau^\varepsilon_D \leq kT + r_0\} \cap \{X^\varepsilon_{\tau^\varepsilon_D} \in N\}$ for positive integers k. We see that

$$\begin{aligned}
\mathbb{P}_{x_0}(A) &\leq \mathbb{P}_{x_0}(X^\varepsilon_{r_0} \notin B_{2\varrho}(x_{\text{stable}})) + \sup_{y\in S_{2\varrho},\, s\geq r_0} \mathbb{P}_{s,y}(\tau^\varepsilon_D \leq kT) \\
&\leq \mathbb{P}_{x_0}(X^\varepsilon_{r_0} \notin B_{2\varrho}(x_{\text{stable}})) + \sum_{m=0}^{k} \mathbb{P}_{s,y}(\tau^\varepsilon_D = \tau_m,\, \xi^{s,y}_{\tau^\varepsilon_D} \in N) \\
&\qquad + \mathbb{P}_{s,y}\left(\min_{1\leq m\leq k}(\vartheta_m - \tau_{m-1}) \leq T\right) \\
&\leq \mathbb{P}_{x_0}(X^\varepsilon_{r_0} \notin B_{2\varrho}(x_{\text{stable}})) + (2k+1)e^{-(\overline{Q}_\infty(N)-\eta/2)/\varepsilon}.
\end{aligned}$$

The choice $k = \lfloor (e^{(\overline{Q}_\infty(N)-\eta)/\varepsilon} - r_0)/T \rfloor + 1$ yields

$$\mathbb{P}_{x_0}(A) \leq \mathbb{P}_{x_0}(X^\varepsilon_{r_0} \notin B_{2\varrho}(x_{\text{stable}})) + 5T^{-1}e^{-\eta/2\varepsilon}.$$

This implies that $\mathbb{P}_{x_0}(\tau^\varepsilon_D \leq e^{(\overline{Q}_\infty(N)-\eta)/\varepsilon},\, X^\varepsilon_{\tau^\varepsilon_D} \in N) \to 0$ as $\varepsilon \to 0$. Now choose η small enough such that $\overline{Q}_\infty(N) - \eta > \overline{Q}_\infty + \eta$. Then Proposition 10.7 states that the exit time of the domain D is smaller than $e^{(\overline{Q}_\infty+\eta)/\varepsilon}$ with probability close to 1. The combination of these two results implies $\mathbb{P}_{x_0}(X^\varepsilon_{\tau^\varepsilon_D} \in N) \to 0$ as $\varepsilon \to 0$. $\square$

Chapter 11

Examples

This final chapter presents some examples that illustrate the effect of self-stabilization on the exit behavior. In the gradient case, which allows for explicit calculations of quasi-potentials, we shall see that self-stabilization may change the picture completely.

11.1 The gradient case

The structural assumption on Φ, namely its rotational invariance as stated in (8.13), implies that Φ is always a potential gradient. In fact, this assumption means that Φ is the gradient of the positive potential

$$\mathcal{A}(x) = \int_0^{\|x\|} \phi(u)du.$$

In this chapter, we make the additional assumption that the second drift component given by the vector field V is also a potential gradient, which brings us back to the very classical situation of gradient type time homogeneous Itô diffusions. In this situation, quasi-potentials and exponential exit rates may be computed rather explicitly and allow for a good illustration of the effect of self-stabilization on the asymptotics of exit times.

We assume from now on that $V = -\nabla U$ is the gradient of a potential U on $\mathbb{R}^d$. Then the drift of the limiting diffusion Y^∞ defined by (9.13) is also a potential gradient, that is

$$b(x) := V(x) - \Phi(x - x_{\text{stable}}) = -\nabla(U(x) + \mathcal{A}(x - x_{\text{stable}})).$$

In this setting, we have according to Lemma 3.2 (see also Theorem 4.3.1 in [19])

$$Q^\infty(x_{\text{stable}}, z) = 2(U(z) - U(x_{\text{stable}}) + \mathcal{A}(z - x_{\text{stable}}))$$

for all $z \in \overline{D}$ that are 'seen' by the quasi-potential. In particular,

$$\overline{Q}_\infty = 2 \inf_{z \in \partial D} (U(z) - U(x_{\text{stable}}) + \mathcal{A}(z - x_{\text{stable}})).$$

Observe that the exit time of the self-stabilizing diffusion is strictly larger than that of the classical diffusion defined by

$$dZ_t^\varepsilon = V(Z_t^\varepsilon)dt + \sqrt{\varepsilon} dW_t, \quad Z_0^\varepsilon = x_0.$$

Indeed, by Kramers' law (Theorem 3.1), the corresponding exit times satisfy

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_0}(\tau_D^\varepsilon(Z^\varepsilon)) = \inf_{z \in \partial D} 2(U(z) - U(x_{\text{stable}})) < \overline{Q}_\infty = \lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_{x_0}(\tau_D^\varepsilon(X^\varepsilon)).$$

In fact, the exit behavior is completely different if we compare the diffusions with resp. without self-stabilization. The above formulas already indicate that the effective additional potential $\mathcal{A}$ may totally change the geometry. Let us next have a closer look at this matter by comparing exit locations.

11.2 The general one-dimensional case

In this section we confine ourselves to one-dimensional self-stabilizing diffusions. In dimension one, the structural conditions concerning Φ and V (Assumption 8.2) are always granted, and we may study the influence of self-stabilization on exit laws in a general setting.

Let $a < 0 < b$, and assume for simplicity that the unique stable equilibrium point is the origin. Denote by $U(x) = -\int_0^x V(u)du$ the potential that induces the drift V. As seen before, the interaction drift is the gradient of the potential $\mathcal{A}(x) = \int_0^{|x|} \phi(u)du$. Since we are in the gradient situation, the exponential rate for the mean exit time from the interval $[a, b]$ can be computed as follows.
If we denote by $\tau_x(X^\varepsilon) = \inf\{t \geq 0 : \ X_t^\varepsilon = x\}$ the first passage time of the level x for the process X^ε and $\tau_I = \tau_a \wedge \tau_b$, then the exit law of the classical diffusion Z^ε (i.e. without self-stabilization) is described by

$$\lim_{\varepsilon \to 0} \mathbb{P}_0 \left(e^{(Q_0^\infty - \eta)/\varepsilon} < \tau_I(Z^\varepsilon) < e^{(Q_0^\infty + \eta)/\varepsilon} \right) = 1,$$

and

$$\lim_{\varepsilon \to 0} \varepsilon \log \mathbb{E}_0(\tau_I(Z^\varepsilon)) = Q_0^\infty,$$

where $Q_0^\infty = 2 \min\{U(a), U(b)\}$. Moreover, if we assume that $U(a) < U(b)$, then $\mathbb{P}_0(\tau_I(Z^\varepsilon) = \tau_a(Z^\varepsilon)) \to 1$ as $\varepsilon \to 0$.

The picture changes completely if we introduce self-stabilization. The quasi-potential becomes

$$Q_1^\infty = 2\min\{U(a) + \mathcal{A}(a), U(b) + \mathcal{A}(b)\} > Q_0^\infty,$$

which once again, and this time even more explicitly, shows that the mean exit time of X^ε from the interval I is strictly larger than that of Z^ε. This result corresponds to what intuition suggests: the process needs more work and consequently more time to exit from a domain if it is attracted by some law concentrated around the stable equilibrium point. Furthermore, if a and b satisfy

$$\mathcal{A}(b) - \mathcal{A}(a) < U(a) - U(b),$$

we observe that $\mathbb{P}_0(\tau_I(X^\varepsilon) = \tau_b(X^\varepsilon)) \to 1$, i.e. the diffusion exits the interval at the point b. Thus, we observe the somehow surprising behavior that self-stabilization changes the exit location from the left to the right endpoint of the interval.

11.3 An example in the plane

In this section we give another explicit example in dimension two, in order to illustrate changes of exit locations in more detail.

Let $V = -\nabla U$, where

$$U(x,y) = 6x^2 + \tfrac{1}{2}y^2,$$

and let us examine the exit problem for the elliptic domain

$$D = \{(x,y) \in \mathbb{R}^2 :\ x^2 + \tfrac{1}{4}y^2 < 1\}.$$

The unique stable equilibrium point is the origin $x_{\text{stable}} = 0$.

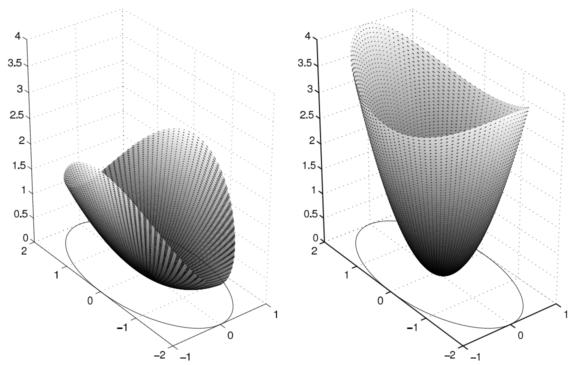

Figure 11.1: Potentials U (l.) and $U + \mathcal{A}$ (r.)

The asymptotic mean exit time of the diffusion Z_t^ε starting in 0 is given by $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_0(\tau_D^\varepsilon(Z^\varepsilon)) = 4$, since the minimum of the potential on ∂D is reached if $y = \pm 2$ and $x = 0$. Let us now focus on its exit location, and denote $N_{(x,y)} = \partial D \cap B_\varrho((x,y))$ for some sufficiently small $\varrho > 0$. The diffusion exits asymptotically in the neighborhood $N_{(0,2)}$ with probability close to 1/2 and in the neighborhood $N_{(0,-2)}$ with the same probability.

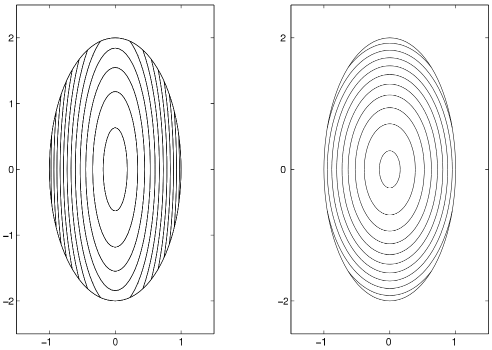

Figure 11.2: Potential level sets without (l.) and with interaction (r.)

Now we look how self-stabilization changes the picture. For the interaction drift we choose

$$\Phi(x,y) = \nabla \mathcal{A}(x,y), \qquad \text{with } \mathcal{A}(x,y) = 2x^2 + 2y^2$$

Firstly, the effect of self-stabilization delaying the exit time is quantified as follows. For the self-stabilizing diffusion X^ε starting in 0 we have $\lim_{\varepsilon\to 0} \varepsilon \log \mathbb{E}_0(\tau_D^\varepsilon(X^\varepsilon)) = 16$.

More demonstrative, though, is the change of exit locations. The diffusion X^ε exits asymptotically with probability close to 1/2 in the neighborhoods $N_{(-1,0)}$ and $N_{(1,0)}$, respectively. This is illustrated by the difference of the level sets of the underlying potentials in Figure 11.2.

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Appendix A

Supplementary tools

The following tools are well known without exception. We state the exponential martingale inequality that was used in Chapter 2, and give a suitable version of Gronwall's lemma.

A.1 Proposition (Exponential martingale inequality)**.** *Let W be a d-dimensional Brownian motion on $(\Omega, \mathscr{F}, \mathbb{P})$, and suppose $\sigma : \mathbb{R}_+ \times \Omega \to \mathbb{R}^{d\times d}$ is continuous, bounded by $m > 0$ and adapted w.r.t. the filtration of W. Let $X_t = \int_0^t \sigma_s dW_s$. Then for each $\delta > 0$ and $T > 0$*

$$\mathbb{P}\left[\sup_{0\le t\le T} \|X_t\| \ge \delta\right] \le 2d \, \exp\left\{-\frac{\delta^2}{2dm^2T}\right\}.$$

Proof. Apply Doob's maximal inequality to the exponential martingale

$$\exp\left\{\alpha \int_0^t \sigma_s dW_s - \frac{\alpha^2}{2}\int_0^t \|\sigma_s\|^2 \, ds\right\},$$

where $\alpha > 0$, and then choose the optimal α to obtain the desired bound. $\square$

A.2 Lemma (Gronwall)**.** *Suppose $f, \phi : [0, T] \to [0, \infty)$ satisfy $\phi, \phi f \in L^1([0, T])$ and*

$$f(t) \le a + \int_0^t \phi(s) f(s) ds, \quad 0 \le t \le T, \tag{A.1}$$

for some constant $a > 0$. Then for $0 \le t \le T$

$$f(t) \le a \, \exp\left\{\int_0^t \phi(s) ds\right\}.$$

Proof. Let $F(t) = \exp\left\{-\int_0^t \phi(s)ds\right\}\int_0^t \phi(s)f(s)ds$. Using the assumption (A.1) on f, we see that $F'(t) \le a\phi(t)\exp\left\{-\int_0^t \phi(s)ds\right\}$. Since $F(0) = 0$, this implies $F(t) \le a\Big(1 - \exp\left\{-\int_0^t \phi(s)ds\right\}\Big)$. Multiplying the latter inequality by $\exp\left\{\int_0^t \phi(s)ds\right\}$ and using (A.1) once again yields the claimed estimate. $\square$

Appendix B

Auxiliary results on large deviations

In this chapter, we state a few results on the problem of diffusion exit from a domain taken from the book by Dembo and Zeitouni [17] that were used in Chapter 10. We consider the family of solutions of the autonomous d-dimensional stochastic differential equation

$$dX_t^\varepsilon = b(X_t^\varepsilon)\,dt + \sqrt{\varepsilon}\,dW_t, \qquad X_0^\varepsilon = x_0 \in \mathbb{R}^d, \tag{B.1}$$

where W is a d-dimensional Brownian motion, and b is assumed to be globally Lipschitz.

Let $D \subset \mathbb{R}^d$ be a bounded domain, and denote by

$$\tau_D^\varepsilon := \inf\Big\{t \geq 0 : \; X_t^\varepsilon \notin D\Big\}$$

the first exit time of X^ε from D.

As in Chapter 3, we assume that D is metastable for the diffusion X^ε by imposing the following conditions.

(i) The deterministic system

$$\dot{\xi} = b(\xi), \quad \xi_0 = x_0, \tag{B.2}$$

possesses a unique stable equilibrium point x^* in D.

(ii) The solutions of (B.2) satisfy

$$\xi_0 \in D \quad \Longrightarrow \quad \xi_t \in D \;\; \forall t > 0, \tag{B.3}$$

$$\xi_0 \in \bar{D} \quad \Longrightarrow \quad \lim_{t\to\infty} \xi_t = x^*. \tag{B.4}$$

This condition is slightly stronger than the one imposed in Chapter 3, since $\lim_{t\to\infty}\xi_t = x^*$ is required also for $\xi_0 \in \partial D$, which excludes the situation of a characteristic boundary.

Let $I_{0t}^{x_0}$ denote the rate function of X^ε given by (1.12), and for $x, y \in \mathbb{R}^d$ define the corresponding quasi-potential

$$V(x,y) = \inf\Big\{I_{0t}^{x}(\varphi) : \varphi \in C_{0t}, \varphi_0 = x, \varphi_t = y, t > 0\Big\}. \tag{B.5}$$

Moreover, assume that the minimal energy required to exit from D is finite, i.e.

$$V^* := \inf_{y\in\partial D} V(x^*, y) < \infty.$$

Then we have the following results.

B.1 Lemma ([17], Lemma 5.7.18)**.** *For any $\eta > 0$ and any $\varrho > 0$ small enough there exists $T_0 > 0$ such that*

$$\liminf_{\varepsilon\to 0} \varepsilon \log \inf_{x_0\in B_\varrho(x^*)} \mathbb{P}_{x_0}\Big(\tau_D^\varepsilon \leq T_0\Big) \;>\; -(V^* + \eta).$$

Let $\varrho > 0$ such that $B_\varrho(x^*) \subset D$, and let

$$\sigma_\varrho = \inf\{t \geq 0 : X_t^\varepsilon \in B_\varrho(x^*) \cup \partial D\}.$$

B.2 Lemma ([17], Lemma 5.7.19)**.** *Then*

$$\lim_{t\to\infty} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x_0\in D} \mathbb{P}_{x_0}\Big(\sigma_\varrho > t\Big) \;=\; -\infty.$$

The following lemma is a slight generalization of Lemma 5.7.21 in [17]. There the closed set N is assumed to be a subset of ∂D. The proof consists of an obvious modification of the original one.

B.3 Lemma. *Let $\tilde D \subset D$ be a domain that contains x^*, let $N \subset \partial\tilde D$ be a closed set, and denote*

$$\tilde\sigma_\varrho = \inf\{t \geq 0 : X_t^\varepsilon \in B_\varrho(x^*) \cup \partial\tilde D\}.$$

Then

$$\lim_{\varrho\to 0} \limsup_{\varepsilon\to 0} \varepsilon \log \sup_{x_0\in\partial B_{2\varrho}(x^*)} \mathbb{P}_{x_0}\Big(X^\varepsilon_{\tilde\sigma^\varepsilon_\varrho} \in N\Big) \;\leq\; -\inf_{z\in N} V(x^*, z).$$

List of notations

$\|\cdot\|$	Euclidean norm on $\mathbb{R}^d$
a_μ	transition time of weakly periodic diffusion in Part II
$B_\varrho(x)$	open ball of radius ϱ centered at x
C_{0T}	space of continuous functions from $[0,T]$ to $\mathbb{R}^d$
$\|\cdot\|_{0T}$	sup norm on C_{0T}
ρ_{0T}	uniform metric on C_{0T}
H^y_{0T}	Cameron-Martin space of a.c. functions on $[0,T]$ starting at y
$\mathbb{P}_x(X \in \cdot)$	law of the diffusion X starting at x
$\mathcal{L}(Y)$	law of the random variable Y
$\mathscr{B}(\mathcal{X})$	Borel subsets of the topological space $\mathcal{X}$
$\mathcal{P}(\mathcal{X})$	space of probability measures on $\mathcal{X}$
$\boldsymbol{\lambda}$	Lebesgue measure
Λ_T	space of dissipative drift functions (Part III)
$\|\cdot\|_T$	norm in Λ_T

Selbständigkeitserklärung

Hiermit erkläre ich, daß ich die vorliegende Dissertation selbständig und ohne unerlaubte Hilfe angefertigt habe. Alle Hilfsmittel und Hilfen habe ich angegeben. In der Einleitung ist erläutert, welche Teile der Arbeit in Kooperation mit anderen Autoren entstanden sind.

Dierk Peithmann